

On the Extension of the RFRS Property

Burns Healy

Tufts University

brendan.healy@tufts.edu

June 25th, 2015

Rational Derived Subgroup

Definition

Let G be a group. We define the *rational derived subgroup*, denoted $G_r^{(1)}$ to be as follows:

$$G_r^{(1)} := \{x \in G \mid \exists k \neq 0 \text{ such that } x^k \in G^{(1)}\}.$$

Recall that $G^{(1)} := [G, G]$ is the commutator subgroup. We note that

$$G^{(1)} \leq G_r^{(1)}$$

Note:

$$G/G_r^{(1)} \cong G^{\text{ab}}/\textit{torsion}$$

The RFRS Condition

Definition

A group G is called *residually finite rationally solvable* or *RFRS* if there exists a sequence of subgroups

$$G = G_0 > G_1 > G_2 > \dots$$

such that

- $G \triangleright G_i$
- $\bigcap_i G_i = \{1\}$
- $[G : G_i] < \infty$
- $G_{i+1} \geq (G_i)_r^{(1)}$

Preliminary Results

After defining the RFRS property, Ian Agol in his 2008 paper about the Virtual Fiberings Conjecture (now theorem) makes two immediate observations about this property.

Fact

The class of groups that are RFRS is closed under direct products.

Fact

The class of groups that are RFRS is closed under free products.

Semi-Direct Products

In the proof of the previous two claims, Agol constructs explicit subgroups for the product from the original subgroup sequences.

Question

For what semi-direct group products can we construct a subgroup sequence exhibiting the RFRS property?

Lemma

In pursuing this question, we find the following observation helpful.

Lemma (H.)

For any finitely generated group G with the RFRS property, there exists a sequence of subgroups that are all characteristic in G that witness the RFRS property.

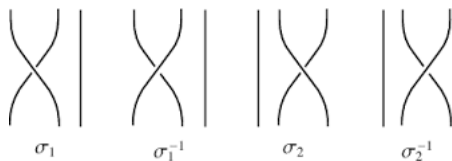
A Partial Result

Theorem (H.)

Let G, H be groups that are RFRS with respect to characteristic subgroup sequences $\{G_i\}, \{H_i\}$ such that $\phi|_{H_i} : H_i \rightarrow \text{Aut}(G_i)$ has the property that, for all $h \in H_i$, the induced action of ϕ_h on G_i^{ab} is trivial. Then $G \rtimes_{\phi} H$ has the RFRS property. Furthermore, the sequence $G_i \rtimes H_i$ is a witnessing sequence.

Why care about these hypotheses?

Braid groups are a highly studied class of groups in geometric group theory. Elements of the n -strand braid group are ways to configure n strings that fix the starting position set-wise.



A finite presentation for this group was discovered by Artin.

$$B_n = \langle \sigma_i \ (1 \leq i < n) \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_i \quad (1 \leq i < n-1), \\ [\sigma_i, \sigma_j] \quad \text{if } |i-j| \geq 2 \end{array} \rangle$$

Why care about these hypotheses?

The pure braid groups are a finite-index subgroup of braid groups, where strands are required to return to their original position at the end of the braid.

$$1 \rightarrow P_n \hookrightarrow B_n \twoheadrightarrow \Sigma_n \rightarrow 1$$

Why care about these hypotheses?

A paper by Cohen and Suciu exhibits an interesting property of braid groups. They split into a semidirect product of free groups.

$$P_n \cong F_{n-1} \rtimes_{\alpha_{n-1}} P_{n-1}$$
$$P_n \cong F_{n-1} \rtimes_{\alpha_{n-1}} (F_{n-2} \rtimes_{\alpha_{n-2}} (F_{n-3} \cdots \rtimes_{\alpha_2} \mathbb{Z}) \cdots)$$

Automorphism Maps

These maps are initially defined as mapping from the whole braid group, but in this context we can consider their restriction to the pure braid subgroup.

Fact

All elements of the image $im_{\alpha_i}(P_i)$ have the property that their action on the corresponding free group acts by conjugating each generator by some word (this word may differ for each generator, and also may be trivial).

What's next

All this combines to say that, if one can find a subgroup sequence of an arbitrary free group that is a RFRS sequence and inherits the property that the induced automorphism action is trivial on its abelianization is trivial, this will show that pure braid groups are RFRS, and that braid groups are virtually RFRS.

Thank you for your attention!
Questions?