COMMENSURABILITY AND QUASI-ISOMETRIC CLASSIFICATION
OF HYPERBOLIC SURFACE GROUP AMALGAMS

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Abstract. Let $\mathcal{C}_S$ denote the class of groups isomorphic to the fundamental group of two closed hyperbolic surfaces identified along an essential simple closed curve in each. We construct a bi-Lipschitz map between the universal covers of these spaces equipped with a CAT($-1$) metric, proving all groups in $\mathcal{C}_S$ are quasi-isometric. The class $\mathcal{C}_S$ has infinitely many abstract commensurability classes, which we characterize in terms of the ratio of the Euler characteristic of the two surfaces and the topological type of the curves identified. We characterize the groups in $\mathcal{C}_S$ that contain a maximal element in their abstract commensurability class restricted to $\mathcal{C}_S$.

1. INTRODUCTION

To study finitely generated infinite groups, one may use algebraic and geometric classifications. Abstract commensurability defines an algebraic equivalence relation on the class of groups, where we say two groups are abstractly commensurable if they contain isomorphic subgroups of finite index. The geometry of finitely generated groups may be studied by identifying a group with a family of metric spaces, which is well-defined up to quasi-isometric equivalence and includes all Cayley graphs for the group constructed with respect to finite generating sets. A finitely generated group is quasi-isometric to any finite-index subgroup, so, if two finitely generated groups are abstractly commensurable, then they are quasi-isometric. Two fundamental questions in geometric group theory are to classify the abstract commensurability and quasi-isometry classes within a class of finitely generated groups and to understand for which classes of groups the characterizations coincide. In this paper, we present a solution to these questions restricted to the class $\mathcal{C}_S$ of groups isomorphic to the fundamental group of two closed hyperbolic surfaces identified along an essential simple closed curve in each.

Our first theorem gives the quasi-isometric classification.

Theorem 1. If $X_1 = S_{g_1} \cup_{\gamma} S_{g_1'}$ and $X_2 = S_{g_2} \cup_{\rho} S_{g_2'}$ where $S_{g_i}$ and $S_{g_i'}$ are closed hyperbolic surfaces for $i = 1, 2$ and $\gamma : S^1 \to X_1$ and $\rho : S^1 \to X_2$ are the images of essential simple closed curves under identification, then there exists a bi-Lipschitz equivalence $\Phi : \widetilde{X_1} \to \widetilde{X_2}$ mapping lifts of $\gamma$ to lifts of $\rho$ bijectively.

Corollary 2. If $G_1, G_2 \in \mathcal{C}_S$, then $G_1$ and $G_2$ are quasi-isometric.
Corollary 2 also follows from Theorem 4.14 of Malone’s unpublished work in [7]; groups in $C_S$ are examples of geometric amalgamations of free groups with the same degree refinement, a matrix associated to the graph of groups splitting. The proof in [7] follows the techniques of Behrstock and Neumann in [1]: a model space is constructed by identifying the universal cover of a surface with boundary with a fattened tree and gluing these trees along boundary curves. The bi-Lipschitz equivalence of these spaces follows from [1], and thus these groups are shown to be quasi-isometric. In our result, we study the CAT($-1$) structure of these groups, constructing an explicit bi-Lipschitz equivalence between the universal covers of two closed hyperbolic surfaces identified along essential simple closed curves. The class $C_S$ is a subclass of the class of limit groups, described in [2]; in particular, all groups in $C_S$ are constructible limit groups of level 1. The quasi-isometric classification within the class of limit groups in full generality remains open, and we expect the techniques of this paper may be used towards this classification.

The class $C_S$ is also a subclass of the class $C$ of groups isomorphic to the fundamental group of two hyperbolic surfaces identified along a primitive closed curve in each. We prove that within $C$, any group quasi-isometric to a group in $C_S$ is in $C_S$.

**Proposition 3.** If $G \in C_S$ and $G' \in C \setminus C_S$, then $G$ and $G'$ are not quasi-isometric.

The abstract commensurability classes within $C_S$ are finer than the quasi-isometry classes; while there is a unique quasi-isometry class in $C_S$, there are infinitely many abstract commensurability classes. Whyte, in [14], proves a similar result for free products of hyperbolic surface groups, which may be thought of as fundamental groups of closed hyperbolic surfaces identified along null-homotopic curves in each, and are also a subclass of limit groups.

**Theorem 4.** ([14], Theorem 1.6, 1.7) Let $\Sigma_g$ be the fundamental group of a surface of genus $g \geq 2$ and let $m, n \geq 2$. Let $\Gamma_1 \cong \Sigma_{a_1} \ast \Sigma_{a_2} \ast \cdots \ast \Sigma_{a_n}$ and $\Gamma_2 \cong \Sigma_{b_1} \ast \Sigma_{b_2} \ast \cdots \ast \Sigma_{b_m}$. Then $\Gamma_1$ and $\Gamma_2$ are quasi-isometric, and $\Gamma_1$ and $\Gamma_2$ are abstractly commensurable if and only if
\[
\frac{\chi(\Gamma_1)}{n - 1} = \frac{\chi(\Gamma_2)}{m - 1}.
\]

Our classification in $C_S$ up to abstract commensurability depends on Euler characteristic as well. Our techniques rely on the results of [4], [6] that the class of groups $C_S$ and its class of finite-index subgroups are topologically rigid: any isomorphism between subgroups of finite index in this class is induced by a homeomorphism between $K(G,1)$ spaces that consist of a set of closed surfaces identified along a set of essential simple closed curves. We prove the ratio of the Euler characteristic of the surfaces identified and the topological type of the curves identified are obstructions to the existence of homeomorphic finite sheeted covers, and we construct a common cover in the absence of these obstructions. Our result is the following.

**Theorem 5.** If $G_1, G_2 \in C_S$, then $G_1$ and $G_2$ are abstractly commensurable if and only if $G_1$ and $G_2$ may be expressed as $\pi_1(S_{g_1}) \ast_{(a_1)} \pi_1(S_{g_1}')$ and $\pi_1(S_{g_2}) \ast_{(a_2)} \pi_1(S_{g_2}')$, respectively.
respectively, so that \( a_i \) identifies \( \gamma_i \in \pi_1(S_g) \) and \( \gamma_i' \in \pi_1(S_g') \), and the following conditions hold.

(a) \( \frac{\chi(S_g)}{\chi(S_g')} = \frac{\chi(S_g')}{\chi(S_g''')} \),

(b) \( t(\gamma_1) = t(\gamma_2) \),

(c) \( t(\gamma_1') = t(\gamma_2') \),

where \( t(\gamma) = 1 \) if \( \gamma \) is a non-separating simple closed curve, and \( t(\gamma) = \frac{\chi(S_r)}{\chi(S_s)} \) if \( \gamma \) is a separating simple closed curve such that \( S_g = S_r \cup S_s \) and \( \chi(S_r) \leq \chi(S_s) \).

The result in Theorem 5 is related to the abstract commensurability classification of the right-angled Coxeter groups considered by Crisp and Paoluzzi in [4],

\[ W_{m,n} = W(\Gamma_{m,n}) \]

where \( \Gamma_{m,n} \) denotes the graph consisting of a circuit of length \( m + 4 \) and a circuit of length \( n + 4 \) which are identified along a common subpath of edge-length 2. For all \( m \) and \( n \), the group \( W_{m,n} \) is the orbifold fundamental group of a 2-dimensional reflection orbi-complex \( O_{m,n} \) which is finitely covered by a space consisting of two hyperbolic surfaces identified along a single non-separating essential simple closed curve. Conversely, all amalgams of surface groups over non-separating essential simple closed curves are finite index subgroups of \( W_{m,n} \) for some \( m \) and \( n \), dependent on the Euler characteristic of the two surfaces. Thus, our theorem extends their result.

**Corollary 6.** ([4] Theorem 1.1) Let \( 1 \leq m \leq n \) and \( 1 \leq k \leq \ell \). Then \( W_{m,n} \) and \( W_{k,\ell} \) are abstractly commensurable if and only if \( \frac{m}{n} = \frac{k}{\ell} \).

Dani and Thomas, in [5], prove a quasi-isometric classification within a class of right-angled Coxeter groups which includes \( W_{m,n} \), and our result in Theorem 1 also gives the following.

**Corollary 7.** ([5] Theorem 1.5) For all \( m, n, k, \ell \geq 1 \), \( W_{m,n} \) and \( W_{k,\ell} \) are quasi-isometric.

In order to extend the result of Crisp and Paoluzzi to the setting of separating simple closed curves, we prove the following result.

**Proposition 8.** Let \( G_1, G_2 \in \mathcal{C}_S \). If \( G_1 \) and \( G_2 \) are abstractly commensurable then there exist normal subgroups \( N_1 \triangleleft G_1 \) and \( N_2 \triangleleft G_2 \) of finite index such that \( N_1 \cong N_2 \).

A **maximal element** in the abstract commensurability class of a group \( G \) is a group \( G_0 \) that contains every other element in the abstract commensurability class of \( G \) as a finite-index subgroup. A classic result in this setting is that of Margulis [8], who proved that if \( H \leq \text{PSL}(2, \mathbb{C}) \) is a discrete subgroup of finite covolume, then there exists a maximal element in the abstract commensurability class of \( H \) if and only if \( H \) is non-arithmetic. It follows that the commensurability class of a non-arithmetic finite-volume hyperbolic 3-manifold contains a **minimal element**: there exists an orbifold finitely covered by every other manifold in the commensurability class. Recent surveys
on notions of commensurability are given by Paoluzzi [10] and Walsh [13]. Within the class $C_S$, we prove the following analog.

Define the maximal element in the abstract commensurability class of $G \in C_S$ restricted to $C_S$ to be a group $G_0 \in C_S$ that contains every other element in the abstract commensurability class of $G$ in $C_S$ as a finite index subgroup. We prove that a maximal element restricted to the class $C_S$ exists only under the following conditions.

**Proposition 9.** Let $G \cong \pi_1(S_g) *_{\langle \gamma \rangle} \pi_1(S_{g'}) \in C_S$ be given by the monomorphisms $\gamma \mapsto [\gamma_g] \in \pi_1(S_g)$, and $\gamma \mapsto [\gamma_{g'}] \in \pi_1(S_{g'})$. Then there exists a maximal element in the abstract commensurability class of $G$ restricted to $C_S$ if and only if $\gamma_g$ and $\gamma_{g'}$ are separating simple closed curves with $S_g = S_{r,1} \cup \gamma_g S_{s,1}$, $S_{g'} = S_{r',1} \cup \gamma_{g'} S_{s',1}$ and $r \neq s$, $r' \neq s'$.

The outline of the paper is as follows. The space and groups we consider are defined in Section 2. In Section 3, we describe the geometry and quasi-isometry classes of groups in $C_S$; we construct a bi-Lipschitz map between the universal covers of specified $K(G,1)$ spaces for any $G \in C_S$ and prove the quasi-isometry class containing $C_S$ in $C$ does not contain any group not in $C_S$. In Section 4, we describe the topology of a set of $K(G,1)$ spaces and their finite-sheeted covers, prove the abstract commensurability classification, and characterize the groups in $C_S$ that contain a maximal element in their abstract commensurability class restricted to $C_S$.

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## 2. Preliminaries

An orientable surface of genus $g$ with $b$ boundary components will be denoted $S_{g,b}$; we use $S_g$ to denote $S_{g,0}$. A hyperbolic surface is a surface that admits a metric of constant negative curvature. A (hyperbolic) surface group refers to the fundamental group of a (hyperbolic) surface. A closed curve in a surface $S_{g,b}$ is a continuous map $S^1 \to S_{g,b}$, and we typically identify a closed curve with its image in the surface. An essential closed curve $\gamma$ is primitive if it cannot be written as $[\gamma] = [\gamma_0^n]$ for some closed curve $\gamma_0$. A closed curve is simple if it is embedded. A homotopy class of simple closed curves is a homotopy class in which there exists a simple closed curve representative. A multicurve in $S_{g,b}$ is the union of a finite collection of disjoint simple closed curves in $S_{g,b}$.

Let $\mathcal{X}$ denote the class of spaces homeomorphic to two hyperbolic surfaces identified along a primitive closed curve in each. Let $\mathcal{X}_S \subset \mathcal{X}$ be the subclass in which the
curves that are identified are simple. Let \( C \) be the class of groups isomorphic to the fundamental group of a space in \( \mathcal{X} \), and let \( C_S \subset C \) be the subclass of groups isomorphic to the fundamental group of a space in \( \mathcal{X}_S \). If \( G \in C \) then \( G \cong \pi_1(S_g) \ast_{\gamma} \pi_1(S_h) \), the amalgamated free product of two hyperbolic surface groups over \( \mathbb{Z} \). We suppress in our notation the monomorphisms \( i_g : \langle \gamma \rangle \to \pi_1(S_g) \) and \( i_h : \langle \gamma \rangle \to \pi_1(S_h) \) given by \( i_g : \gamma \mapsto [\gamma_g], \ i_h : \gamma \mapsto [\gamma_h] \), where \( \gamma_g : S^1 \to S_g \) and \( \gamma_h : S^1 \to S_h \).

3. Quasi-isometric classification

3.1. The hyperbolic structure of groups in \( C_S \). Let \( G \in C_S \) so that \( G \cong \pi_1(X) \), where \( X = S_g \cup \gamma S_h \subset \mathcal{X}_S \), and \( \gamma \) is the image of \( \gamma_g : S^1 \to S_g \) identified to \( \gamma_h : S^1 \to S_h \) in \( X \). One can choose hyperbolic metrics on \( S_g \) and \( S_h \) so that the length of the geodesic representatives of \( [\gamma_g] \) and \( [\gamma_h] \) is equal. Gluing by an isometry yields a piecewise hyperbolic complex with \( \text{CAT}(-1) \) universal cover \( \tilde{X} \). For details on gluing constructions, see ([3], Section II.11.) The space \( \tilde{X} \) consists of convex subspaces of \( \mathbb{H}^2 \) that are the preimages of hyperbolic surfaces with boundary, identified along geodesic lines that are the preimages of the curve \( \gamma \). The universal cover \( \tilde{X} \) is a geometric graph of spaces in the following sense. For more details on graphs of groups and graphs of spaces, see [11], [12].

**Definition 10.** A geometric graph of spaces is a graph of spaces, \( G \), consisting of a set of vertex spaces, \( \{V_i\}_{i \in I} \), and a set of edge spaces, \( \{E_{i,j}\}_{(i,j) \in J} \), with \( J \subset I \times I \), \( i < j \), so that the vertex and edge spaces are geodesic metric spaces and there are isometric embeddings

\[
E_{i,j} \to V_i, \\
E_{i,j} \to V_j,
\]

as convex subsets. The geometric realization of \( G \) is the metric space \( X \) given by the disjoint union of the vertex and edge spaces, identified according to the relations of \( G \), and given the induced path metric. The underlying graph of the graph of spaces \( G \) is the abstract graph specifying \( G \).

To prove that all groups in \( C_S \) are quasi-isometric, we exhibit a bi-Lipschitz equivalence between the universal covers of the corresponding spaces in \( \mathcal{X}_S \).

**Definition 11.** A map \( f : (X, d_X) \to (Y, d_Y) \) is \( K \)-bi-Lipschitz if there exists \( K \geq 1 \) with

\[
\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)
\]

and a \( K \)-bi-Lipschitz equivalence if, in addition, it is a homeomorphism. A map is said to be a bi-Lipschitz equivalence if it is a \( K \)-bi-Lipschitz equivalence for some \( K \). Two spaces \( X \) and \( Y \) are bi-Lipschitz equivalent if there exists a bi-Lipschitz equivalence from \( X \) to \( Y \).

We will need the following two lemmas.
Lemma 12. Suppose $P$ and $P'$ are compact convex hyperbolic polygons. Then $P$ and $P'$ are bi-Lipschitz equivalent.

Lemma 13. Let $T_1$ and $T_2$ be tilings of convex regions $R_1, R_2 \subset \mathbb{H}^2$ by compact convex polygons of finitely many isometry types. If $T_1$ and $T_2$ have isomorphic dual graphs, then $R_1$ and $R_2$ are bi-Lipschitz equivalent.

Proof. The tilings $T_1$ and $T_2$ of convex subsets of $\mathbb{H}^2$ are examples of locally finite geometric graphs of spaces, where the vertex spaces are the tiles, the underlying graph is the dual graph to the tiling, and $R_1$ and $R_2$ are the geometric realizations of $T_1$ and $T_2$, respectively. In this language, suppose $T_1$ and $T_2$ consist of a set of vertex and edge spaces $\{V_i\}_I, \{E_{i,j}\}_J$ and $\{W_i\}_I, \{F_{i,j}\}_J$, respectively. Since $T_1$ and $T_2$ have isomorphic dual graphs and there are finitely many isometry types of polygons, by Lemma 12, there exists a $K$-bi-Lipschitz equivalence $\phi_i : V_i \to W_i$ for all $i \in I$ such that $\phi_i(E_{i,j}) = \phi_j(E_{i,j}) = F_{i,j}$ for all $(i,j) \in J$. There is an induced bijection $\Phi : R_1 \to R_2$ defined by $\Phi|_{V_i} = \phi_i(V_i)$, which is continuous by the pasting lemma.

Let $x, y \in R_1$, and let $p$ be the geodesic path from $x$ to $y$. The path $p$ can be decomposed into a union of geodesic segments $\{[x_i, x_{i+1}]\}_{i=0}^{n-1}$, with $x_0 = x$ and $x_n = y$ so that the interior of each subpath $[x_i, x_{i+1}]$ is contained entirely in a vertex space $V_i$ since there are finitely many isometry types of compact polygons in the tiling. If a segment is contained in an edge space $E_{i,j}$, we may choose either $V_i$ or $V_j$ since $\phi_i(E_{i,j}) = \phi_j(E_{i,j})$. By assumption, there exists a $K$-bi-Lipschitz equivalence $\phi_i : V_i \to W_i$ for all $i$. Since $\Phi(p)$ is a path connecting $\Phi(x)$ and $\Phi(y)$,

$$d(\Phi(x), \Phi(y)) = \sum_{i=0}^{n-1} d(\phi_i(x_i), \phi_i(x_{i+1}))$$

$$\leq \sum_{i=0}^{n-1} Kd(x_i, x_{i+1})$$

$$= K \sum_{i=0}^{n-1} d(x_i, x_{i+1})$$

$$= Kd(x, y).$$

The other inequality follows similarly. Namely, suppose $q$ is a geodesic path from $\Phi(x)$ to $\Phi(y)$. The path $q$ can be decomposed into a union of geodesic segments $\{[w_i, w_{i+1}]\}_{i=0}^{m-1}$ where $w_0 = \Phi(x), w_m = \Phi(y)$ and the interior of $[w_i, w_{i+1}]$ is contained
entirely in a vertex space, $W_i$. Then, since $\Phi^{-1}(q)$ is a path from $x$ to $y$, and $\phi_i$ is a $K$-bi-Lipschitz equivalence for all $i$,

$$d(\Phi(x), \Phi(y)) = \sum_{i=0}^{m-1} d(w_i, w_{i+1})$$

$$\geq \sum_{i=0}^{m-1} \frac{1}{K} d(\phi_i^{-1}(w_i), \phi_i^{-1}(w_{i+1}))$$

$$= \frac{1}{K} \sum_{i=0}^{m-1} d(\phi_i^{-1}(w_i), \phi_i^{-1}(w_{i+1}))$$

$$\geq \frac{1}{K} d(x, y).$$

Thus, $\frac{1}{K} d(x, y) \leq d(\Phi(x), \Phi(y)) \leq K d(x, y)$, so $\Phi$ is a $K$-bi-Lipschitz equivalence. \qed

3.2. Bi-Lipschitz equivalence of spaces $\tilde{X}$ with $X \in \mathcal{X}_S$. In this section, we construct a bi-Lipschitz equivalence between spaces $\tilde{X}$ and $\tilde{X}'$ with $X, X' \in \mathcal{X}_S$ in two steps. In Proposition 14 we construct a bi-Lipschitz equivalence between connected hyperbolic regions in the complement of the set of lifts in the universal covers of $X$ and $X'$, giving an explicit map from the boundary lifts of one region onto the boundary lifts in another region. Then, in Theorem 15, we extend this map to the entire universal covers, $\tilde{X}$ and $\tilde{X}'$.

**Proposition 14.** Let $S_g$ and $S_h$ be closed hyperbolic surfaces each equipped with a hyperbolic metric. Let $\gamma : S^1 \to S_g$ and $\rho : S^1 \to S_h$ be essential simple closed curves. Let $L_\Gamma$ be the set of lifts of $\gamma$ in $\mathbb{H}^2$ and let $L_\rho$ be the set of lifts of $\rho$ in $\mathbb{H}^2$. If $R_\gamma$ is the closure of a component of $\mathbb{H}^2 \setminus L_\Gamma$ and $R_\rho$ is the closure of a component of $\mathbb{H}^2 \setminus L_\rho$, then there exists a bi-Lipschitz equivalence

$$\tilde{\phi} : R_\gamma \to R_\rho,$$

mapping the lifts bounding $R_\gamma$ bijectively onto the lifts bounding $R_\rho$. 

**Figure 1.** On the left are lifts of a generating curve for a genus two surface, and on the right, lifts of a curve representing the commutator element. Both figures were drawn with Curt McMullen’s lim program [9].
Proof. Let $L_\gamma$ be the set of lifts bounding $R_\gamma$ and $L_\rho$ be the set of lifts bounding $R_\rho$. If $\tilde{\gamma}$ denotes the axis of $[\gamma]$ and $\tilde{\rho}$ denotes the axis of $[\rho]$, we may assume $\tilde{\gamma} \in L_\gamma$ and $\tilde{\rho} \in L_\rho$ since $\pi_1(S_g)$ and $\pi_1(S_h)$ act transitively on the set of lifts.

The outline of the proof is as follows. We first restrict to specific hyperbolic metrics on $S_g$ and $S_h$ and specify a generating set $S$ for $\pi_1(S_g)$ and a generating set $T$ for $\pi_1(S_h)$. We define a map between the lifts,

$$\phi : L_\gamma \rightarrow L_\rho,$$

recursively, as a union of bijective partial functions, $\phi_n$. The recursion hypothesis is if $|g|_S \leq n$, where $|g|_S$ denotes the word length of $g$ with respect to the generating set $S$, then $g\tilde{\gamma} \in L_\gamma$ is included in the domain of $\phi_n$ and if $|h|_T \leq n$, then $h\tilde{\rho} \in L_\rho$ is included in the image of $\phi_n$. Thus, the map $\phi$ is exhaustive on $L_\gamma$ and $L_\rho$.

The construction of $\phi$ induces a tiling of the regions $R_\gamma$ and $R_\rho$ by compact convex hyperbolic polygons and a homeomorphism

$$\bar{\phi} : R_\gamma \rightarrow R_\rho.$$

Note that this tiling is determined by $\phi$ and is different from the tiling of $\mathbb{H}^2$ by fundamental domains; in general, the tiling contains more than one isometry type of tile and is not preserved by $\pi_1(S_g)$. We prove there are finitely many isometry types of tiles in $R_\gamma$ and $R_\rho$ and the dual graph to each tiling is a tree. We show $\bar{\phi}$ maps tiles in $R_\gamma$ to tiles in $R_\rho$ and induces an isomorphism between the dual trees. Therefore, we may apply Lemma 13 to conclude that $R_\gamma$ and $R_\rho$ are bi-Lipschitz equivalent.

To make these notions precise, we begin by restricting to specific hyperbolic metrics on $S_g$ and $S_h$ and giving the non-standard generating sets $S$ and $T$, defined with respect
Figure 3. The generating set \( S \) is defined so that \( d_S(g, h) = 1 \) since \( g \) and \( h \) lie in fundamental domains that share a vertex.

The generating set \( S \) is defined so that \( d_S(g, h) = 1 \) since \( g \) and \( h \) lie in fundamental domains that share a vertex.

to these metrics and chosen to make the recursion hypotheses hold. The following, when described for \( \gamma \) and \( \pi_1(S_g) \), holds likewise for \( \rho \) and \( \pi_1(S_h) \).

Any two hyperbolic metrics on \( S_g \) are bi-Lipschitz equivalent, so assume the hyperbolic metric on \( S_g \) is given by a representation \( \pi_1(S_g) \to PSL(2, \mathbb{R}) \) with a fundamental domain for the action of \( \pi_1(S_g) \) on the hyperbolic plane a \( 4g \)-gon and so that the axis of \( [\gamma]\), \( \tilde{\gamma} : \mathbb{R} \to \mathbb{H}^2 \), intersects each translate of the fundamental domain in a single geodesic segment. Parametrize \( \tilde{\gamma} \) so that \( \tilde{\gamma}(n) \) is the midpoint of a geodesic segment in a translate of the fundamental domain for all \( n \in \mathbb{Z} \). Since \( \pi_1(S_g) \) acts on \( \mathbb{H}^2 \) by isometries, \( g\tilde{\gamma}(n) \) is the midpoint of a geodesic segment of \( g\tilde{\gamma} \) for all \( g \in \pi_1(S_g) \) and all \( n \in \mathbb{Z} \). Give the set of lifts a cell structure by declaring \( g\tilde{\gamma}(n) \) a 0-cell for all \( g \in \pi_1(S_g) \) and \( n \in \mathbb{Z} \).

Let \( S \) be the generating set for \( \pi_1(S_g) \) that consists of the \( 16g^2 - 8g \) elements of \( \pi_1(S_g) \) that map \( \tilde{\gamma}(0) \) onto the fundamental domains that share a vertex with the fundamental domain containing \( \tilde{\gamma}(0) \). Let \( d_S \) be the word metric on \( \pi_1(S_g) \) with generating set \( S \). Since there is a one-to-one correspondence between 0-cells in \( \mathbb{H}^2 \) and elements of \( \pi_1(S_g) \), define \( d_S(g\tilde{\gamma}(s), g_0\tilde{\gamma}(r)) \) to be \( d_S(g\tilde{\gamma}^s, g_0\tilde{\gamma}^r) \).

The map between the set of lifts must preserve the ordering on the boundary, so orient \( \partial_\infty \mathbb{H}^2 \), the visual boundary of \( \mathbb{H}^2 \). View \( \partial_\infty \mathbb{H}^2 \cong S^1 \) as the unit interval \([0, 1]\) with the identification \( 0 \sim 1 \). A geodesic line, \( \ell(t) \), defines two points on the boundary, which we denote \( \ell^+ \) and \( \ell^- \). Orient the boundary clockwise so that \( \hat{\gamma}^+ = 0 \). We may assume that for all lifts \( g\tilde{\gamma} \in L_\gamma \), \( \partial_\infty(g\tilde{\gamma}) \subset S^1 \setminus [0, \hat{\gamma}^-] \). Since \( \gamma \) and \( \rho \) are simple closed curves, the order on the boundary gives a total order on the lifts in \( L_\gamma \) and \( L_\rho \); we say \( g\tilde{\gamma} < h\tilde{\gamma} \) if \( \partial_\infty(g\tilde{\gamma}) < \partial_\infty(h\tilde{\gamma}) \).

Construction:

In this section, we specify a map between the lifts, \( \phi : L_\gamma \to L_\rho \), and a map between the regions \( \phi : R_\gamma \to R_\rho \), recursively, with \( \phi = \bigcup_n \phi_n \) and \( \phi = \bigcup_n \phi_n \). The maps \( \phi_n \)
induce a tiling of $R_\gamma$ and $R_\rho$ by compact convex hyperbolic polygons and the map $\tilde{\phi}_n : R_\gamma \to R_\rho$. The recursion hypotheses are:

1. If $|g|_S \leq n$, then $g\tilde{\gamma} \in L_\gamma$ is included in the domain of $\phi_n$, and if $|h|_T \leq n$, then $h\tilde{\rho} \in L_\rho$ is included in the image of $\phi_n$.

2. Each tile has at most $M = 3 + \max\{64g^2 - 32g, 64h^2 - 32h\}$ edges, and each edge constructed in the interior of $R_\gamma$ connects vertices $g_0\tilde{\gamma}(r)$ and $g\tilde{\gamma}(s)$ with $d_S(g_0\tilde{\gamma}(r), g\tilde{\gamma}(s)) = 1$.

3. The dual graph to each tiling is a tree, and $\tilde{\phi}$ maps tiles in $R_\gamma$ to tiles in $R_\rho$, inducing an isomorphism between the dual trees.

Stage one:

Begin by choosing the set of lifts of $\gamma$ and $\rho$ in the domain and image of $\phi_1$. Let $W_\gamma^1$ and $W_\rho^1$ be the set of all vertices in $R_\gamma$ and $R_\rho$ at distance 1 from $\tilde{\gamma}(0)$ and $\tilde{\rho}(0)$ in the metrics $d_S$ and $d_T$, respectively, and select and label every lift that intersects these sets as follows. No lift intersects two edges of any fundamental domain that share a vertex, so a lift intersects $W_\gamma^1$ in at most three vertices, $\{g_{i-1}\tilde{\gamma}, g_i\tilde{\gamma}, g_{i+1}\tilde{\gamma}\}$ for some $i \in \mathbb{Z}$, as illustrated in Figure 4.

Let

\[
S_\gamma^1 = \{g_1\tilde{\gamma}(0), \ldots, g_m\tilde{\gamma}(0)\} \subset W_\gamma^1
\]

\[
S_\rho^1 = \{h_1\tilde{\rho}(0), \ldots, h_n\tilde{\rho}(0)\} \subset W_\rho^1
\]

be the subset of vertices of $W_\gamma^1$ and $W_\rho^1$ so that for $S_\gamma^1$:

a) One vertex of each lift that intersects $W_\gamma^1$ is included.

b) If a lift intersects $W_\gamma^1$ in three vertices, the middle vertex is chosen.

c) The vertices are labeled so that $g_i\tilde{\gamma} < g_j\tilde{\gamma}$ if $i < j$.

Likewise for $S_\rho^1$. Condition (a) ensures recursion hypothesis (1) is satisfied, and condition (b) ensures $d_S(g_i\tilde{\gamma}(0), g_{i+1}\tilde{\gamma}(0)) = 1$ and $d_T(h_i\tilde{\rho}(0), h_{i+1}\tilde{\rho}(0)) = 1$, and condition (c) is used to preserve the ordering on the boundary.

If $m = n$, continue to the definition of $\phi_1$. Otherwise, if $m < n$, let $W_\gamma^{1,r}$ be the set of all vertices in $R_\gamma$ at distance 1 from $\{\tilde{\gamma}(1), \tilde{\gamma}(2), \ldots, \tilde{\gamma}(r)\}$, and select the $n - m$ lifts that intersect $W_\gamma^{1,r}$ lowest in the ordering from the boundary. If there do not exist $n - m$ such lifts that intersect $W_\gamma^{1,r}$, increase $r$. Choose and label one vertex of each of these lifts so that if a lift intersects $W_\gamma^{1,r}$ in three vertices, the middle vertex is chosen, and the selected vertices are labeled $g_{m+1}\tilde{\gamma}(0), \ldots, g_n\tilde{\gamma}(0)$ so that $g_i\tilde{\gamma} < g_j\tilde{\gamma}$ if $i < j$. Then $d_S(g_i\tilde{\gamma}(0), g_{i+1}\tilde{\gamma}(0)) = 1$ for $1 \leq i < n$. 
We define $\phi_1: g_i \tilde{\gamma} \mapsto h_i \tilde{\rho}$; to give the map explicitly, preserving the orientation on the boundary, re-label the lifts by

$$L_1^\gamma = \{ \ell_{e_g}(t), \ell_{g_1}(t), \ldots, \ell_{g_n}(t) \} \subset L_\gamma$$
$$L_1^\rho = \{ \ell_{e_h}(t), \ell_{h_1}(t), \ldots, \ell_{h_n}(t) \} \subset L_\rho$$

where $\ell_{g_i} = g_i \tilde{\gamma}$, $e_g$ and $e_h$ denote the identity elements, and $\ell_{g_i}$ is parametrized so that $\ell_{g_i}(0) = g_i \tilde{\gamma}(0)$, $d_S(\ell_{g_i}(p), \ell_{g_i}(q)) = |p - q|$ for $p, q \in \mathbb{Z}$, and $\ell_{g_i}^{+\infty} < \ell_{g_i}^{-\infty}$; likewise for $L_1^\rho$. Define

$$\phi_1: L_1^\gamma \to L_1^\rho$$

by,

$$\phi_1(\ell_{e_g}(t)) = \ell_{e_h}(t),$$
$$\phi_1(\ell_{g_i}(t)) = \ell_{h_i}(t)$$

for all $1 \leq i \leq n$.

If $m \leq n$, construct a hyperbolic polygon, $P_1^\gamma$, in $R_\gamma$ with vertices

$$V_1^\gamma = \{ \tilde{\gamma}(-1), g_1 \tilde{\gamma}(0), \ldots, g_n \tilde{\gamma}(0), \tilde{\gamma}(r) \},$$

edges connecting vertices adjacent in the list, and an edge connecting $\tilde{\gamma}(r)$ and $\tilde{\gamma}(-1)$, where $r$ is given above if $m < n$ and $r = 1$ if $m = n$. Similarly, construct a hyperbolic polygon, $P_1^\rho$, in $R_\rho$ with vertices

$$V_1^\rho = \{ \tilde{\rho}(-1), h_1 \tilde{\rho}(0), \ldots, h_n \tilde{\rho}(0), \tilde{\rho}(1) \},$$

edges connecting vertices adjacent in the list, and an edge connecting $\tilde{\rho}(1)$ and $\tilde{\rho}(-1)$. If $m > n$, the process is carried out in reverse, and the final vertices in the sets $V_1^\gamma$ and $V_1^\rho$ are $\tilde{\gamma}(1)$ and $\tilde{\rho}(s)$ for some $s$. Define

$$\bar{\phi}_1: P_1^\gamma \mapsto P_1^\rho$$
by mapping the vertices in \( V^1_\gamma \) to the vertices in \( V^1_\rho \), in order, and extending to a homeomorphism on the edges and interior.

The polygons \( P^1_\gamma \) and \( P^1_\rho \) have an equal number of sides, \( M \), by construction. We claim

\[
3 \leq M \leq 2 + \max\{16g^2 - 8g, 16h^2 - 8h\}.
\]

By the choice of representation, each translate of the fundamental domain that intersects \( R_\gamma \) intersects \( R_\rho \) in at least two vertices, so \( m, n \geq 1 \), proving the lower bound. On the other hand, \( m, n \leq \max\{16g^2 - 8g, 16h^2 - 8h\} \) by the choice of generating sets, proving the upper bound. In addition, each edge constructed in the interior of \( R_\gamma \) satisfies \( d_S(g_i \tilde{\gamma}(0), g_{i+1} \tilde{\gamma}(0)) = 1 \) for \( 1 \leq i \leq n - 1 \). Likewise for \( P^1_\rho \). The dual tree after stage one is a single point, thus, all recursion hypotheses are satisfied.

**Stage \( n \):**

Suppose \( \phi_{n-1}, n-1 \) stages of the tiling, and \( \bar{\phi}_{n-1} \) have been constructed, satisfying the hypothesis of the recursion. For each of the \( K \) edges constructed in the interior of \( R_\gamma \) during stage \( n-1 \), extend the maps \( \phi_{n-1} \) and \( \bar{\phi}_{n-1} \) and the tiling as follows.

Suppose \( \{g_0 \tilde{\gamma}(0), g\tilde{\gamma}(0)\} \) is the \( i^{th} \) edge constructed during stage \( n-1 \) in the interior of \( R_\gamma \), \( g_0 \tilde{\gamma} < g\tilde{\gamma} \), and \( \phi_{n-1} \) induces the map \( \phi_{n-1} : \{g_0 \tilde{\gamma}(0), g\tilde{\gamma}(0)\} \mapsto \{h_0 \tilde{\rho}(0), h\tilde{\rho}(0)\} \).

**Figure 5.** A sketch of the lifts mapped during stage 1, and, below, the first two polygons constructed in the tilings of \( R_\gamma \) and \( R_\rho \).
Consider

\[
\begin{align*}
(1) & \quad \{ g_0 \tilde{\gamma}(-1), g_0 \tilde{\gamma}(0), g \tilde{\gamma}(0), g \tilde{\gamma}(1) \}, \\
(2) & \quad \{ h_0 \tilde{\rho}(-1), h_0 \tilde{\rho}(0), h \tilde{\rho}(0), h \tilde{\rho}(1) \}.
\end{align*}
\]

In a manner similar to that of stage 1, first choose the lifts of \( \gamma \) and \( \rho \) in the domain and image of \( \phi_{n_i} \). Let \( W_{\gamma}^{n_i} \) and \( W_{\rho}^{n_i} \) be the set of all vertices in \( R_\gamma \) and \( R_\rho \) at distance 1 from (1) and (2) that do not lie on a lift included in the domain or image of \( \phi_{n-1} \), respectively. Select and label each lift that intersects one of these sets as follows. Let

\[
S_{\gamma}^{n_i} = \{ g_{i,1} \tilde{\gamma}(0), \ldots, g_{i,r} \tilde{\gamma}(0) \} \subset W_{\gamma}^{n_i}
\]

\[
S_{\rho}^{n_i} = \{ h_{i,1} \tilde{\rho}(0), \ldots, h_{i,s} \tilde{\rho}(0) \} \subset W_{\rho}^{n_i}
\]

be the subset of vertices of \( W_{\gamma}^{n_i} \) and \( W_{\rho}^{n_i} \) chosen so that for \( S_{\gamma}^{n_i} \):

(a) One vertex of each lift that intersects \( W_{\gamma}^{n_i} \) is included.

(b) If a lift intersects \( W_{\gamma}^{n_i} \) in three vertices, the middle vertex is chosen.

(c) The vertices are labeled so that \( g_{i,k} \tilde{\gamma} < g_{j,k} \tilde{\gamma} \) if \( i < j \).

Likewise for the vertices in \( S_{\rho}^{n_i} \). Condition (b) ensures \( d_S(g_{i,k} \tilde{\gamma}(0), g_{i,k+1} \tilde{\gamma}(0)) = 1 \) and \( d_T(h_{i,k} \tilde{\rho}(0), h_{i,k+1} \tilde{\rho}(0)) = 1 \).

If \( r = s \), continue to the definition of \( \phi_{n_i} \). Otherwise, if \( r < s \), let \( W_{\gamma}^{n_i,k} \) be the set of all vertices in \( R_\gamma \) at distance 1 from \( \{ g \tilde{\gamma}(2), g \tilde{\gamma}(3), \ldots, g \tilde{\gamma}(k) \} \) that do not lie on a lift included in the domain of \( \phi_{n-1} \), and select the \( s - r \) lifts that intersect \( W_{\gamma}^{n_i,k} \) lowest in the ordering from the boundary. If there do not exist \( s - r \) such lifts, increase \( k \). Choose and label one vertex of each of these lifts so that if a lift intersects \( W_{\gamma}^{n_i,k} \)
in three vertices, the middle vertex is chosen, and the vertices selected are labeled $g_{i,r+1}\tilde{\gamma}(0), \ldots, g_{i,s}\tilde{\gamma}(0)$ so that $g_{i,j}\tilde{\gamma} < g_{i,k}\tilde{\gamma}$ if $j < k$. Then $d_S(g_{i,j}\tilde{\gamma}(0), g_{i,j+1}\tilde{\gamma}(0)) = 1$ for $1 \leq j < s$.

As before, we map $g_{i,j}\tilde{\gamma}$ to $h_{i,j}\tilde{\rho}$; to define the map $\phi_{n_i}$, label these lifts

$$L_{\gamma}^{n_i} = \{\ell_{g_{i,1}}(t), \ldots, \ell_{g_{i,s}}(t)\} \subset L_{\gamma},$$

$$L_{\rho}^{n_i} = \{\ell_{h_{i,1}}(t), \ldots, \ell_{h_{i,s}}(t)\} \subset L_{\rho}$$

where $\ell_{g_{i,k}} = g_{i,k}\tilde{\gamma}$ and is parametrized so that $\ell_{g_{i,k}}(0) = g_{i,k}\tilde{\gamma}(0), d_S(\ell_{g_{i,k}}(p), \ell_{g_{i,k}}(q)) = |p - q|$ for $p, q \in \mathbb{Z}$ and $\ell_{g_{i,k}}^{+\infty} < \ell_{g_{i,k}}^{-\infty}$, likewise for $L_{\rho}^{n_i}$. Define

$$\phi_{n_i} : L_{\gamma}^{n_i} \to L_{\rho}^{n_i}$$

by

$$\phi_{n_i}(\ell_{g_{i,j}}(t)) = \ell_{h_{i,j}}(t)$$

for all $1 \leq j \leq s$ and let

$$\phi_n = \phi_{n-1} \cup \bigcup_{i=1}^{K} \phi_{n_i}.$$ 

If $r \leq s$, construct a hyperbolic polygon, $P_{\gamma}^{m_i}$ in $R_{\gamma}$ with vertices

$$V_{\gamma}^{m_i} = \{g_0\tilde{\gamma}(0), g_0\tilde{\gamma}(-1), g_{i,1}\tilde{\gamma}(0), \ldots, g_{i,s}\tilde{\gamma}(0), g_{i,1}\tilde{\gamma}(0), \ldots, g_0\tilde{\gamma}(0)\},$$
edges connecting adjacent vertices in the list, and an edge connecting $g_0\gamma(0)$ and $g\gamma(0)$, where $k$ is given above if $r < s$ and $k = 2$ if $r = s$. Construct a hyperbolic polygon, $P_{\rho}^{n_i}$ in $R_\rho$ with vertices

$$V_{\rho}^{n_i} = \{h_0\bar{\rho}(0), h_0\bar{\rho}(-1), h_{i,1}\bar{\rho}(0), \ldots, h_{i,s}\bar{\rho}(0), h\bar{\rho}(2), h\bar{\rho}(1), h\bar{\rho}(0)\},$$

edges connecting vertices adjacent in the list, and an edge connecting $h_0\bar{\rho}(0)$ and $h\bar{\rho}(0)$. If $r > s$, the process is carried out in reverse. Define

$$\tilde{\phi}_{n_i} : P_{\gamma}^{n_i} \mapsto P_{\rho}^{n_i}$$

by mapping vertices in $V_{\gamma}^{n_i}$ to vertices in $V_{\rho}^{n_i}$ in the order given in the lists, and extending to the edges and interior by a homeomorphism. Let

$$\tilde{\phi}_n = \tilde{\phi}_{n-1} \cup \bigcup_{i=1}^{K} \tilde{\phi}_{n_i}.$$  

To prove the first recursion hypothesis holds after stage $n$, consider $\Gamma$, the Cayley graph for $\pi_1(S_g)$ with generating set $S$, which maps into the hyperbolic plane with the 0-cells, $\{\gamma(0)\}$, previously described and edges that are geodesic segments. A more standard Cayley graph, $\Gamma'$, for $\pi_1(S_g)$ maps into $\mathbb{H}^2$ dual to the tiling by 4$g$-gons; there is a 4$g$-gon in $\Gamma'$ about each vertex in the tiling. The Cayley graph $\Gamma$ is obtained from the Cayley graph $\Gamma'$ by replacing each 4$g$-gon with $K_{4g}$, the complete graph on 4$g$ vertices. Let $\Gamma_R$ be the subgraph of $\Gamma$ induced by vertices contained in $R_\gamma$. That is, $\Gamma_R$ consists of all vertices that lie on the lifts in $L_\gamma$ and all the edges in $\Gamma$ between these vertices. The subgraph $\Gamma_R$ is connected since any fundamental domain that intersects $R_\gamma$ intersects $R_n$ in at least two vertices. Now, let $g\gamma \in R_\gamma$ with $|g|_S = n$. Since the subgraph $\Gamma_R$ is connected, there exists $g' \in \Gamma_R$ with $d_S(g\gamma(0), g'\gamma(0)) = 1$ and $|g'|_S \leq n - 1$. By the recursion hypothesis, $g'\gamma$ is in the domain of $\phi_{n-1}$, so $g'\gamma(0)$ lies on the edge of a polygon in the interior of $R_\gamma$. Hence $g\gamma$ is in the domain of $\phi_n$ by the choice of elements in the set (1) and $W_{\gamma}^{n_i,k}$.

The bound on the number of sides of the polygons $P_{\gamma}^{n_i}$ and $P_{\rho}^{n_i}$ holds by the choice of elements in $W_{\gamma}^{n_i,1}$, $W_{\rho}^{n_i,1}$ and the size of the generating sets, which imply

$$r \leq 64g^2 - 32g \quad \text{and} \quad s \leq 64h^2 - 32h.$$  

Each edge of $P_{\gamma}^{n_i}$ in the interior of $R_\gamma$ satisfies

$$d_S(g_{i,j}\gamma(0), g_{i,j+1}\gamma(0)) = 1, \quad d_S(h_{i,j}\bar{\rho}(0), h_{i,j+1}\bar{\rho}(0)) = 1$$

for $1 \leq j \leq n - 1$ by construction.

Finally, the dual tree associated to $\tilde{\phi}_{n_i}$ has one additional edge, connecting the tile constructed in stage $n-1$ containing $g_0\gamma(0)$ and $g\gamma(0)$ to the tile constructed associated to $\phi_{n_i}$. Since $\tilde{\phi}$ still maps the edge $\{g_0\gamma(0), g\gamma(0)\}$ to the edge $\{h_0\bar{\rho}(0), h\bar{\rho}(0)\}$, $\tilde{\phi}$ extends the isomorphism between the dual trees.
Finitely many isometry types of tiles:

By this construction, $R_\gamma$ and $R_\rho$ are tiled by compact convex hyperbolic polygons and $\phi$ induces an isomorphism between the dual trees of the tilings. The final claim needed to conclude the proof is that there are only finitely many isometry types of tiles constructed.

By the recursion hypothesis, each polygon in $R_\gamma$ has at most $M$ edges, where $M = 3 + \max\{64g^2 - 32g, 64h^2 - 32\}$. To bound the edge lengths, observe that all but possibly one edge of the polygon connects vertices $v$ and $w$ with $d_S(v, w) = 1$. There are at most $16g^2 - 8g$ edge lengths of these edges; let $L$ be the length of the longest. The last edge connects $g\gamma(k)$ and $g\tilde{\gamma}(1)$ and $k \leq 64h^2 - 32h$. So the maximum length of this edge is $k \times d$, where $d$ is the translation length of $[\gamma]$. Let $N = \max\{L, n \times d\}$.

Let $P$ be a polygon constructed in the tiling and fix a vertex $g\tilde{\gamma}(0)$ in the polygon. All other vertices lie in $B_{g\tilde{\gamma}(0)}(r)$, a ball of radius $r = M \times N$ in the hyperbolic plane with center $g\tilde{\gamma}(0)$. The element $g^{-1} \in \pi_1(S_g)$ is an isometry mapping $g\tilde{\gamma}(0)$ to $\tilde{\gamma}(0)$, and $B_{g\tilde{\gamma}(0)}(r)$ to $B_{\tilde{\gamma}(0)}(r)$, a ball of radius $r$ about $\tilde{\gamma}(0)$. Since the group action preserves the 0-cells, $g^{-1}$ maps $P$ to a polygon in $B_{\tilde{\gamma}(0)}(r)$ with vertices $g_i\tilde{\gamma}(0)$ and $d_S(\tilde{\gamma}(0), g_i\tilde{\gamma}(0)) \leq r$. There are only finitely many polygons in this ball with vertices $g_i\tilde{\gamma}(0)$, so there are only finitely many isometry types of polygons constructed in the tiling. A similar argument holds for the tiling of $R_\rho$, concluding the proof of Proposition 14.

To prove the entire universal covers of $X_1, X_2 \in X_S$ are bi-Lipschitz equivalent, we map the regions incident to the axis of $[\gamma]$ to the regions incident to the axis of $[\rho]$ using the bi-Lipschitz equivalence from Proposition 14. We use the action of the groups by deck transformations on the universal covers to extend the map recursively along regions incident to the lifts that bound a region already in the domain.

**Theorem 15.** If $X_1 = S_{g_1} \cup_\gamma S'_{g_1}$ and $X_2 = S_{g_2} \cup_\rho S'_{g_2}$ where $S_{g_i}$ and $S'_{g_i}$ are closed hyperbolic surfaces for $i = 1, 2$ and $\gamma : S^1 \to X_1$ and $\rho : S^1 \to X_2$ are the images of essential simple closed curves under identification, then there exists a bi-Lipschitz equivalence $\Phi : \tilde{X_1} \to \tilde{X_2}$ mapping lifts of $\gamma$ to lifts of $\rho$ bijectively.

**Proof.** Let $\tilde{\gamma}$ and $\tilde{\rho}$ be the axes for $[\gamma]$ and $[\rho]$, respectively. Then $\tilde{\gamma}$ is contained in a hyperbolic plane stabilized by $\pi_1(S_{g_1})$ and a hyperbolic plane stabilized by $\pi_1(S'_{g_1})$, and $\tilde{\rho}$ is contained in a hyperbolic plane stabilized by $\pi_1(S_{g_2})$ and a hyperbolic plane stabilized by $\pi_1(S'_{g_2})$. Let $R_1$ and $R_2$ be the convex regions bounded by lifts incident to $\tilde{\gamma}$ in the plane stabilized by $\pi_1(S_{g_1})$ and let $R_3$ and $R_4$ be the convex regions bounded by lifts incident to $\tilde{\gamma}$ in the plane stabilized by $\pi_1(S'_{g_1})$. Similarly, let $S_1$ and $S_2$ be the convex regions bounded by lifts incident to $\tilde{\rho}$ in the plane stabilized by $\pi_1(S_{g_2})$ and let $S_3$ and $S_4$ be the convex regions bounded by lifts incident to $\tilde{\rho}$ in the plane stabilized by $\pi_1(S'_{g_2})$. 

By Proposition 14, there exist $K_i$-bi-Lipschitz equivalences
\[ \bar{\phi}_i : R_i \to S_i \]
that are bijective on the lifts bounding $R_i$ and $S_i$ and each map $\tilde{\gamma}$ to $\tilde{\rho}$. Let $K = \max\{K_i\}$ so that $\phi_i$ is a $K$-bi-Lipschitz equivalence for $1 \leq i \leq 4$. Define
\[ \Phi|_{R_i} = \bar{\phi}_i. \]

To extend the map coherently along the regions incident to a lift already included in the domain, color the convex regions bounded by lifts in $\tilde{X}_1$ and $\tilde{X}_2$ with colors $\{C_1, C_2, C_3, C_4\}$ as follows. Let $C(R)$ denote the color of region $R$. Two-color the plane stabilized by $\pi_1(S_{g_1})$ with $C_1$ and $C_2$ so that $C(R_i) = C_i$ and so that adjacent regions have distinct colors. Do the same for the planes stabilized by $\pi_1(S_{g'_1}), \pi_1(S_{g_2}),$ and $\pi_1(S_{g'_2})$ so that $C(R_i) = C(S_i) = C_i$. Extend the coloring to a coloring of $\tilde{X}_1$ and $\tilde{X}_2$ recursively as follows. Suppose a hyperbolic plane, $H$, stabilized by a conjugate of $\pi_1(S_{g_1})$ or $\pi_1(S_{g'_1})$ has not been colored and that $H$ intersects, in a lift $g\tilde{\gamma}$, a hyperbolic plane that has been colored. Then $g^{-1}H$ contains $\tilde{\gamma}$, and has been colored. For every region $R \subset H$, let $C(R) = C(g^{-1}R)$. Color $\tilde{X}_2$ in analogously.

Observe that, regardless of the topological type of $\gamma_i$ and $\gamma'_i$, $\pi_1(X_i)$ is transitive on the regions restricted to one color in $\tilde{X}_1$ and $\tilde{X}_2$.

Define the map $\Phi$ recursively. Suppose $R$ is a region that intersects, in a lift $g\tilde{\gamma}$, the boundary of a region already mapped by $\Phi$ to $\tilde{X}_2$. Then $\Phi : g\tilde{\gamma} \mapsto g'\tilde{\rho}$ for some $g' \in \pi_1(X_2)$. Let $S$ be the region in $\tilde{X}_2$ intersecting $g'\tilde{\rho}$ with $C(R) = C(S) = C_i$ for some $i$, $1 \leq i \leq 4$. Then, map $R$ to $S$ by the map
\[ \Phi|_R = g' \circ \phi_i \circ g^{-1}, \]
a $K$-bi-Lipschitz equivalence since $g$ and $g'$ are isometries. \( \square \)

### 3.3. Amalgams over immersed curves.
All groups in $C_S$ are quasi-isometric, and the quasi-isometric classification of groups of the form $\pi_1(S_g) *_{\langle a \rangle} \pi_1(S_h)$ in full generality depends on the image of $a$ in the groups $\pi_1(S_g)$ and $\pi_1(S_h)$.

**Proposition 16.** If $G \in C_S$ and $G' \in C\setminus C_S$, then $G$ and $G'$ are not quasi-isometric.

**Proof.** All groups in $C$ are $\delta$-hyperbolic, so any quasi-isometry between groups in this class induces a homeomorphism between their visual boundaries. The boundary of a lift of the curves identified is the intersection of two circles in the boundary of the group, $\partial G$. The number of components in the complement of the boundary of a lift depends on whether the curves identified have simple representatives on the surface. That is, if both curves identified are simple, there are four components in the complement of the boundary of a lift in $\partial G$, while if at least one curve has no simple representative, there are fewer than four components in the complement of the boundary of a lift in $\partial G$. A homeomorphism preserves the number of components in the complement of the boundary of a lift, so the fundamental group of a space obtained by identifying closed
hyperbolic surfaces along two essential simple closed curves is not quasi-isometric to the fundamental group of a space obtained by identifying along at least one curve with no simple representative.

**Corollary 17.** If \(G \in \mathcal{C}_S\) and \(G' \in \mathcal{C} \setminus \mathcal{C}_S\), then \(G\) and \(G'\) are not abstractly commensurable.

The quasi-isometric and abstract commensurability classification of groups in \(\mathcal{C}\) remains open.

4. Abstract commensurability classification

In this section, we give the abstract commensurability classification for groups in \(\mathcal{C}_S\) and characterize the groups in \(\mathcal{C}_S\) that contain a maximal element in their abstract commensurability class restricted to \(\mathcal{C}_S\). We use \(\text{lcm}(a, b)\) to denote the least common multiple of \(a\) and \(b\) and \(\gcd(a, b)\) to denote the greatest common divisor of \(a\) and \(b\).

4.1. The topology of spaces in \(\mathcal{X}_S\). The subgroup structure of an amalgamated product is described in the following theorem of Scott and Wall.

**Theorem 18.** ([11], Theorem 3.7) If \(G \cong A *_C B\) and if \(H \leq G\), then \(H\) is the fundamental group of a graph of groups, where the vertex groups are subgroups of conjugates of \(A\) or \(B\) and the edge groups are subgroups of conjugates of \(C\).

Topologically, Theorem 18 implies that any finite-sheeted cover of the space \(X = S_g \cup \gamma S_h\), where \(\gamma\) is the image of \(\gamma_g : S^1 \to S_g\) and \(\gamma_h : S^1 \to S_h\) under identification, consists of a set of surfaces which cover \(S_g\) and a set of surfaces which cover \(S_h\), identified along multicurves that are the preimages of \(\gamma_g\) and \(\gamma_h\). These covers are examples of simple, thick, 2-dimensional hyperbolic \(P\)-manifolds (see [6], Definition 2.3.) So, the following topological rigidity theorem of Lafont allows us to address the abstract commensurability classification for members in \(\mathcal{C}_S\) from a topological point of view. Corollary 20 also follows from the proof of Proposition 3.1 in [4].

**Theorem 19.** ([6], Theorem 1.2) Let \(X_1\) and \(X_2\) be a pair of simple, thick, 2-dimensional hyperbolic \(P\)-manifolds, and assume that \(\phi : \pi_1(X_1) \to \pi_1(X_2)\) is an isomorphism. Then there exists a homeomorphism \(\Phi : X_1 \to X_2\) that induces \(\phi\) on the level of fundamental groups.

**Corollary 20.** Let \(G, G' \in \mathcal{C}_S\) with \(G \cong \pi_1(X)\), \(G' \cong \pi_1(X')\) and \(X, X' \in \mathcal{X}_S\). Then \(G\) and \(G'\) are abstractly commensurable if and only if \(X\) and \(X'\) have homeomorphic finite-sheeted covering spaces.

In the proof of the classification theorem, we make use of the corollary of the following proposition, which, topologically, implies that if \(X, X' \in \mathcal{X}_S\) and \(\pi_1(X)\) and \(\pi_1(X')\) are abstractly commensurable, then \(X\) and \(X'\) have homeomorphic finite-sheeted regular covering spaces.
**Proposition 21.** Let \( G_1, G_2 \in \mathcal{C}_S \). If \( G_1 \) and \( G_2 \) are abstractly commensurable, then there exist normal subgroups of finite index, \( N_1 \triangleleft G_1 \) and \( N_2 \triangleleft G_2 \), such that \( N_1 \cong N_2 \).

**Proof.** Suppose \( G_1 \) and \( G_2 \) are abstractly commensurable. Let \( X_1, X_2 \in \mathcal{X}_S \) with \( G_1 \cong \pi_1(X_1) \) and \( G_2 \cong \pi_1(X_2) \). By Corollary 20, there exist finite-sheeted covering spaces \( Y_1 \to X_1 \) and \( Y_2 \to X_2 \) that are homeomorphic, \( Y_1 \cong Y_2 \).

The spaces \( X_1 \) and \( X_2 \) may be subdivided into finite simplicial complexes, which lifts to a realization of \( Y_1 \) and \( Y_2 \) as finite simplicial complexes. There is a common refinement of these simplicial realizations, and a simplicial homeomorphism \( Y_1 \to Y_2 \).

Let \( X \) denote the universal cover of \( Y_1 \) and \( Y_2 \), equipped with the simplicial structure.

Let \( \text{Aut}(X) \) denote the group of simplicial automorphisms of \( X \). Then \( G_1, G_2 \leq \text{Aut}(X) \). We claim that these are subgroups of finite index; we will show \( X/G_1 \) is a finite-sheeted cover of \( X/\text{Aut}(X) \), and likewise for \( X/G_2 \). To see this, observe that a simplex modulo its automorphism group has positive area so \( X/\text{Aut}(X) \) has positive area. Since \( X/G_1 \) has finite area, it forms a finite-sheeted cover of \( X/\text{Aut}(X) \); likewise for \( X/G_2 \).

So, \( G_1, G_2 \leq \text{Aut}(X) \) are finite-index subgroups, hence so is \( G_1 \cap G_2 \leq \text{Aut}(X) \). The subgroup \( \bigcap_{g \in \text{Aut}(X)} g(G_1 \cap G_2)g^{-1} \) is then a normal finite index subgroup of both \( G_1 \) and \( G_2 \). \( \square \)

To study finite-sheeted covering spaces, we use the following well-known fact.

**Lemma 22.** If \( X \) is a CW-complex and \( X' \) is a degree \( n \) cover of \( X \), then \( \chi(X') = n\chi(X) \), where \( \chi \) denotes Euler characteristic.

A converse to Lemma 22 holds for hyperbolic surfaces with one boundary component, which we use to construct covers of spaces in \( \mathcal{X}_S \).

**Lemma 23.** For \( g_1 \geq 1 \), if \( \chi(S_{g_2,1}) = n\chi(S_{g_1,1}) \), then \( S_{g_2,1} \) \( n \)-fold covers \( S_{g_1,1} \).

**Proof.** Let

\[
\pi_1(S_{g_1,1}) = \langle a_1, b_1, \ldots, a_{g_1}, b_{g_1} \mid \rangle \cong F_{2g_1}
\]

be presentations for the fundamental groups of \( S_{g_1,1} \). Then, the homotopy class of the boundary element \( \gamma_1 : S^1 \to S_{g_1,1} \) corresponds to the element \( [a_1, b_1] \cdots [a_{g_1}, b_{g_1}] \in \pi_1(S_{g_1,1}) \).

We exhibit \( \pi_1(S_{g_2,1}) \) as an index \( n \) subgroup of \( \pi_1(S_{g_1,1}) \) so that in the corresponding cover, \( \gamma_2 \) has preimage a single curve that \( n \)-fold covers \( \gamma_1 \).

Realize \( \pi_1(S_{g_1,1}) \) as the fundamental group of a wedge of \( 2g_1 \) oriented circles labeled by the generating set. Construct an \( n \)-fold cover of this space as a graph, \( \Gamma' \), on \( n \) vertices labeled \( \{0, \ldots, n-1\} \). For every generator besides \( a_1 \), construct an oriented \( n \)-cycle on the \( n \) vertices with each edge labeled by the generator. Since \( \chi(S_{g_1,1}) \) and
χ(S_{g,1}) are both odd, n must be odd as well. Let \{i, i + 1\} and \{i + 1, 1\} be directed edges labeled by \(a_1\) for \(i < n\) and \(i\) odd. Construct a directed loop labeled \(a_1\) at vertex \{0\}. By construction, \(\Gamma\) covers the wedge of circles given above.

To see that \(\gamma_1\) has a preimage with one component, choose a vertex \(v\) in the graph \(\Gamma\) and consider the edge path \(p\) with edges labeled \((a_1, b_1) \ldots (a_g, b_g))^k\), a preimage of a representative of \(\gamma_1\). Then \(n\) is the smallest non-zero \(k\) for which \(p\) terminates at \(v\).

To see this, note that it suffices to consider the path \(p' = [a_1, b_1]^k\) since every other segment \([a_j, b_j]\) returns to its initial vertex. Starting at vertex \{0\}, observe that the path \([a_1, b_1]^k\) terminates at the vertex labeled

\[
\begin{align*}
2k - 1 & \quad \text{if } 0 < k < \left\lfloor \frac{n}{2} \right\rfloor \mod n \\
2n - 2k & \quad \text{if } \left\lfloor \frac{n}{2} \right\rfloor \leq k < n \mod n \\
0 & \quad \text{if } k = 0 \mod n,
\end{align*}
\]

proving the claim. □
If $\gamma : S^1 \to S_g$ is an essential simple closed curve, then up to a homeomorphism of the surface $S_g$, the curve $\gamma$ is either non-separating or separating so that $S_g = S_{r,1} \cup \gamma S_{s,1}$. We characterize the topological type of $\gamma$ with the following notation.

**Definition 24.** Let $\gamma : S^1 \to S_g$ be an essential simple closed curve. Let $t(\gamma)$, the topological type of $\gamma$, be 1 if $\gamma$ is non-separating and $\chi(S_{r,1}) \leq \chi(S_{s,1})$, where $\chi(S_{r,1}) \leq \chi(S_{s,1})$, if $\gamma$ is separating.

**Definition 25.** If $S_g$ and $S_h$ are closed hyperbolic surfaces, $\gamma$ is a multicurve on $S_g$ and $\rho$ is a multicurve on $S_h$, we say $(S_g, \gamma)$ covers $(S_h, \rho)$ if there exists a covering map $p : S_g \to S_h$ so that $\gamma$ is the full preimage of $\rho$ in $S_g$, $\gamma = p^{-1}(\rho)$.

**Proposition 26.** Let $S_{g_1}$ and $S_{g_2}$ be closed hyperbolic surfaces and $\gamma_1 : S^1 \to S_{g_1}$ and $\gamma_2 : S^1 \to S_{g_2}$ essential simple closed curves. There exists $(S_g, \gamma)$ which covers both $(S_{g_1}, \gamma_1)$ and $(S_{g_2}, \gamma_2)$ if and only if $t(\gamma_1) = t(\gamma_2)$.

**Proof.** Suppose there exists $(S_{g_0}, \gamma_0)$ which covers both $(S_{g_1}, \gamma_1)$ and $(S_{g_2}, \gamma_2)$. Then there exists $(S_g, \gamma)$ a regular cover of $(S_{g_1}, \gamma_1)$ and a cover of $(S_{g_2}, \gamma_2)$.

First suppose that $\gamma_1$ is a non-separating simple closed curve; we claim that $t(\gamma_2) = 1$. Either $\gamma_2$ is non-separating, or $\gamma_2$ is separating so that $S_{g_2} = S_{r,1} \cup \gamma_2 S_{s,1}$. Since the cover of $S_{g_1}$ is regular, $S_g \setminus \gamma$ consists of a set of homeomorphic surfaces with boundary. Color $S_{r,1}$ red and $S_{s,1}$ blue, which lifts to a coloring of the components of $S_g \setminus \gamma$. Let $C_R$ denote the red components of $S_g \setminus \gamma$ and let $C_S$ denote the blue components of $S_g \setminus \gamma$; then, $C_R$ covers $S_{r,1}$ and $C_S$ covers $S_{s,1}$. The colored boundary components of the surfaces in $S_g \setminus \gamma$ come in red-blue pairs that project to $\gamma_2$ under the covering map. Since all components of $S_g \setminus \gamma$ are homeomorphic, there are an equal number of red and blue components. Thus, $\chi(C_R) = \chi(C_S)$. So,

$$\chi(C_S) = \chi(C_R) = n\chi(S_{r,1}) = n\chi(S_{s,1}),$$

and therefore $r = s$, and $t(\gamma_2) = 1$. Likewise, if $\gamma_2$ is non-separating, then $t(\gamma_1) = 1$.

Otherwise, both $\gamma_1$ and $\gamma_2$ are separating simple closed curves with $S_{g_1} = S_{r_1,1} \cup \gamma_2 S_{s_1,1}$ and $S_{g_2} = S_{r_2,1} \cup \gamma_2 S_{s_2,1}$, and $r_i \leq s_i$. Since the cover of $S_{g_1}$ is regular, $S_g \setminus \gamma$ consists of two homeomorphism classes of surfaces with boundary: $S_{g_1,b_a}$ which cover $S_{r_1,1}$ and $S_{g_2,b_a}$ which cover $S_{s_1,1}$. Since there is a homeomorphism of $S_g$ taking any lift of $\gamma_2$ to any other lift of $\gamma_2$, we may assume $S_{g_1,b_a}$ cover $S_{r_2,1}$ and $S_{g_2,b_a}$ cover $S_{s_2,1}$, as well.

Suppose that $S_{g_1,b_a}$ forms a $k_i$ degree cover of $S_{r_1,1}$ and that $S_{g_2,b_a}$ forms an $\ell_i$ degree cover of $S_{s_1,1}$. Further, suppose that $S_g$ contains $K$ subsurfaces homeomorphic to $S_{g_1,b_a}$ and $L$ subsurfaces homeomorphic to $S_{g_2,b_a}$. If $S_g$ is an $n_i$ degree cover of $S_{g_i}$, then

$$n_i = L\ell_i = Kk_i.$$

By Lemma 22,

$$\chi(S_{g_1,b_a}) = k_1\chi(S_{r_1,1}) = k_2\chi(S_{r_2,1}),$$

$$\chi(S_{g_2,b_a}) = k_1\chi(S_{s_1,1}) = k_2\chi(S_{s_2,1}).$$
\[ \chi(S_{g,s}) = \ell_1 \chi(S_{s_1,1}) = \ell_2 \chi(S_{s_2,1}). \]

Combining (4) and (5), we have
\[ \frac{\chi(S_{r_1,1})}{\chi(S_{s_1,1})} = \frac{\chi(S_{r_2,1})}{\chi(S_{s_2,1})}, \]
so \( t(\gamma_1) = t(\gamma_2) \).

For the converse, let \( L = -\lcm(\lvert \chi(S_{g_1}) \rvert, \lvert \chi(S_{g_2}) \rvert) \) and let \( k_i = \frac{L}{\chi(S_{g_i})} \). We construct \((S_g, \gamma)\) so that \( \chi(S_g) = 2L \) and \( \gamma \) has two components, each of which covers \( \gamma_i \) by degree \( k_i \).

First suppose that \( t(\gamma_1) = t(\gamma_2) = 1 \). Let \( S_g \) be the closed surface with Euler characteristic \( 2L \). Let \( \gamma \) be a multicurve on \( S_g \) so that \( S_g \setminus \gamma \) consists of two surfaces with Euler characteristic \( L \) and two boundary components. We prove \((S_g, \gamma)\) covers both \((S_{g_1}, \gamma_1)\); the proof that \((S_g, \gamma)\) covers \((S_{g_2}, \gamma_2)\) is similar. If \( \gamma_1 \) is non-separating, the pair \((S_g, \gamma)\) covers \((S_{g_1}, \gamma_1)\) via an intermediate cover
\[ (S_g, \gamma) \xrightarrow{2} (\widehat{S_{g_1}} \setminus \gamma_1) \xrightarrow{k_1} (S_{g_1}, \gamma_1). \]

Let \( \widehat{S_{g_1}} \) be the closed surface with Euler characteristic \( L \). Arranging the handles symmetrically about a central handle, choose \( \gamma_1 : S^1 \to \widehat{S_{g_1}} \) a non-separating simple closed curve so that upon rotation by \( \frac{2\pi}{k_1} \), \( \gamma_1 \) projects to \( \gamma_1 \) and \((\widehat{S_{g_1}} \setminus \gamma_1)\) is a cyclic cover of \((S_{g_1}, \gamma_1)\) of degree \( k_1 \). Then \((S_g, \gamma)\) covers \((\widehat{S_{g_1}} \setminus \gamma_1)\): arrange the handles symmetrically about the multicurve \( \gamma \) and rotate by \( \pi \).

Otherwise, \( \gamma_1 \) is separating with \( S_{g_1} = S_{r,1} \cup_{\gamma_1} S_{s,1} \) and \( r = s \). If \( g_2 \) is even, the pair \((S_g, \gamma)\) covers \((S_{g_1}, \gamma_1)\) via an intermediate cover
\[ (S_g, \gamma) \xrightarrow{2} (\widehat{S_{g_1}} \setminus \gamma_1) \xrightarrow{k_1} (S_{g_1}, \gamma_1), \]
where \( k_1 = \frac{L}{\chi(S_{g_1})} \). Since \( g_2 \) is even, \( L \) is congruent to \( 2 \) modulo \( 4 \), so \( \frac{L}{2} \) is odd.

To construct \( \widehat{S_{g_1}} \), let \( \widehat{S_{r,1}} \) be the surface with one boundary component and Euler characteristic \( \frac{L}{2} \) and let \( \widehat{S_{s,1}} \) be the surface with one boundary component and Euler characteristic \( \frac{L}{2} \). By Lemma 23, \( \widehat{S_{r,1}} \) covers \( S_{r,1} \) by degree \( k_1 \) and \( \widehat{S_{s,1}} \) covers \( S_{s,1} \) by degree \( k_1 \). Let \( \widehat{S_{g_1}} = \widehat{S_{r,1}} \cup_{\gamma_1} \widehat{S_{s,1}} \) be the surface obtained by identifying \( \widehat{S_{r,1}} \) and \( \widehat{S_{s,1}} \) along their boundary curves. Then \( \gamma_1 \) is a separating curve that projects to \( \gamma_1 \) and \((\widehat{S_{g_1}}, \gamma_1)\) covers \((S_{g_1}, \gamma_1)\) by degree \( k_1 \). To see that \((S_g, \gamma)\) covers \((\widehat{S_{g_1}} \setminus \gamma_1)\), cut \( \widehat{S_{g_1}} \) along a non-separating curve in \( \widehat{S_{r,1}} \) and a non-separating curve in \( \widehat{S_{s,1}} \) and double the resulting surface with four boundary components. Re-gluing the boundary components in pairs yields \((S_g, \gamma)\), as illustrated in Figure 9.

If \( g_2 \) is odd, \((S_g, \gamma)\) covers \((S_{g_1}, \gamma_1)\) by degree \( \frac{\chi(S_{g_1})}{\chi(S_{g_2})} = \frac{2L}{\chi(S_{g_1})} = 2^n d \), where \( n \geq 1 \) and \( d \) is an odd integer. The covering is given via intermediate covers:
\[ (S_g, \gamma) \xrightarrow{2} \ldots \xrightarrow{2} (\widehat{S_{g_1}} \setminus \gamma_1) \xrightarrow{2} (\widehat{S_{g_1}} \setminus \gamma_1) \xrightarrow{d} (S_{g_1}, \gamma_1). \]
The first intermediate cover, \((\widetilde{S}_{g_1}, \gamma_1) \xrightarrow{d} (S_{g_1}, \gamma_1)\), is constructed similarly to above. That is, since \(\chi(S_{g_1})\) is congruent to 2 modulo 4, \(\frac{d\chi(S_{g_1})}{2}\) is odd. If \(d > 1\), let \(\widetilde{S}_{r,1}\) be the surface with one boundary component and Euler characteristic \(\frac{d\chi(S_{g_1})}{2}\) and let \(\widetilde{S}_{s,1}\) be the surface with one boundary component and Euler characteristic \(\frac{d\chi(S_{g_1})}{2}\). By Lemma 23, \(\widetilde{S}_{r,1}\) covers \(S_{r,1}\) by degree \(d\) and \(\widetilde{S}_{s,1}\) covers \(S_{s,1}\) by degree \(d\). Let \(\widetilde{S}_{g_1} = \widetilde{S}_{r,1} \cup \gamma_2 \widetilde{S}_{s,1}\) be the surface obtained by identifying \(\widetilde{S}_{r,1}\) and \(\widetilde{S}_{s,1}\) along their boundary curves. Then \((\widetilde{S}_{g_1}, \gamma_1)\) covers \((S_{g_1}, \gamma_1)\) by degree \(d\).

The second intermediate cover, \((\widetilde{S}_{g_1}, \gamma_1) \xrightarrow{2} (\widetilde{S}_{g_1}, \gamma_1)\), is the degree two cover of \((\widetilde{S}_{g_1}, \gamma_1)\) obtained by cutting \(\widetilde{S}_{g_1}\) along a non-separating curve in \(\widetilde{S}_{r,1}\) and a non-separating curve in \(\widetilde{S}_{s,1}\), doubling the resulting surface with four boundary components, and re-gluing along the boundary curves as illustrated in Figure 9. Then, \(\gamma_1\) is a multicurve that bounds two subsurfaces in \(\widetilde{S}_{g_1}\), each with two boundary components and the same Euler characteristic.

If \(n = 1\), then \((S_g, \gamma) = (\widetilde{S}_{g_1}, \gamma_1)\). Otherwise, \((S_g, \gamma)\) covers \((\widetilde{S}_{g_1}, \gamma_1)\) by degree \(2^{n-1}\); cut \(\widetilde{S}_{g_1}\) along a non-separating curve that intersects each of the components of \(\gamma\) once. Double the resulting surface with two boundary components and re-glue the boundary components in pairs to form a double cover of \((\widetilde{S}_{g_1}, \gamma_1)\). Repeat this process \(n - 1\) times to obtain the cover \((S_g, \gamma) \xrightarrow{2} \ldots \xrightarrow{2} (\widetilde{S}_{g_1}, \gamma_1)\).

Suppose now that \(t(\gamma_1) = t(\gamma_2) \neq 1\) so that \(S_{g_1} = S_{r_1,1} \cup \gamma_1 S_{s_1,1}\), \(S_{g_2} = S_{r_2,1} \cup \gamma_2 S_{s_2,1}\), and \(\chi(S_{r_1,1}) = \chi(S_{s_1,1})\). Construct \((S_g, \gamma)\) via an intermediate cover

\[(S_g, \gamma) \xrightarrow{2} (S_{g_0}, \gamma_0) \xrightarrow{k_1} (S_{g_1}, \gamma_1)\]

Let \(L_R = -\ell cm(|\chi(S_{r_1,1})|, |\chi(S_{s_1,1})|)\) and \(L_S = -\ell cm(|\chi(S_{r_2,1})|, |\chi(S_{s_2,1})|)\); then \(L_R\) and \(L_S\) are both odd integers. Let \(S_{R,1}\) be the surface with one boundary component and Euler characteristic \(L_R\) and let \(S_{S,1}\) be the surface with one boundary component and Euler characteristic \(L_S\). By Lemma 23, \(S_{R,1}\) covers \(S_{r_1,1}\) with degree \(\frac{L_R}{\chi(S_{r_1,1})}\) and
$S_{S,1}$ covers $S_{a,1}$ with degree $\frac{L_S}{x(S_{a,1})}$. Since $\frac{x(S_{S,1},i)}{x(S_{a,1})} = \frac{x(S_{S,2},i)}{x(S_{a,1})}$, $L_B = \frac{L_S}{x(S_{a,1})}$. Let $S_{g_0} = S_{R,1} \cup_{\gamma_0} S_{S,1}$, be the surface obtained by identifying $S_{R,1}$ and $S_{S,1}$ along their boundary curves. Then, $(S_{g_0}, \gamma_0)$ covers $(S_{g_i}, \gamma_i)$ by degree $k_i$.

Let $(S_g, \gamma)$ be the 2-fold cover of $(S_{g_0}, \gamma_0)$ obtained by cutting $S_{g_0}$ along a non-separating curve in $S_{R,1}$ and a non-separating curve in $S_{S,1}$ and doubling the resulting surface with four boundary components. Re-gluing the boundary components in pairs yields $(S_g, \gamma)$. Then $(S_g, \gamma)$ covers $(S_{g_i}, \gamma_i)$ and $\gamma$ is a multicurve with two components, each of which covers $\gamma_i$ by degree $k_i$.

4.2. Abstract commensurability classification. In this section we give the complete abstract commensurability classification within $C_S$. We will use the following notation in the proof of the theorem: if $S_g$ and $S_h$ are closed surfaces and $\gamma_g \subset S_g$ and $\gamma_h \subset S_h$ are (the images of) multicurves on $S_g$ and $S_h$, we write $(S_g, \gamma_g) \cong (S_h, \gamma_h)$ if there exists a homeomorphism $S_g \to S_h$ mapping $\gamma_g$ bijectively to $\gamma_h$.

**Theorem 27.** If $G_1, G_2 \in C_S$, then $G_1$ and $G_2$ are abstractly commensurable if and only if $G_1$ and $G_2$ may be expressed as $G_1 \cong \pi_1(S_{g_1}) *_{(a_1)} \pi_1(S_{g_1}')$ and $G_2 \cong \pi_1(S_{g_2}) *_{(a_2)} \pi_1(S_{g_2}')$, given by the monomorphisms $a_i \mapsto [\gamma_i] \in \pi_1(S_{g_i})$ and $a_i \mapsto [\gamma_i'] \in \pi_1(S_{g_i}')$, and the following conditions hold.

\[
\begin{align*}
(a) \quad & \frac{x(S_{g_1})}{x(S_{g_1}')} = \frac{x(S_{g_2})}{x(S_{g_2}')} , \\
(b) \quad & t(\gamma_1) = t(\gamma_2), \\
(c) \quad & t(\gamma_1') = t(\gamma_2').
\end{align*}
\]

**Proof.** Let $G_1, G_2 \in C_S$ so that $G_1 \cong \pi_1(X_1)$ and $G_2 \cong \pi_1(X_2)$ with $X_1, X_2 \in \mathcal{X}$, and suppose $G_1$ and $G_2$ are abstractly commensurable. By Proposition 21, there exist finite-index normal subgroups $N_1 \lhd G_1$ and $N_2 \lhd G_2$ so that $N_1 \cong N_2$. Let $p_1 : \tilde{X}_1 \to X_1$ and $p_2 : \tilde{X}_2 \to X_2$ be the regular covers corresponding to the subgroups $N_1$ and $N_2$, respectively. By Theorem 19, $\tilde{X}_1 \cong \tilde{X}_2$.

Let $X_1 = S_{g_1} \cup_{a_1} S_{g_1}'$ where $a_1 : S^1 \to X_1$ is the image of the essential simple closed curve $\gamma_1 : S^1 \to S_{g_1}$ and $\gamma_1' : S^1 \to S_{g_1}'$ under identification. Then $G_1 \cong \pi_1(S_{g_1}) *_{(a_1)} \pi_1(S_{g_1}')$ as in the statement of the theorem. Let

\[
S = p_1^{-1}(S_{g_1}, \gamma_1) = \bigsqcup_{i \in I} (S_i, \rho_i) \quad \text{and} \quad S' = p_1^{-1}(S_{g_1}', \gamma_1') = \bigsqcup_{i' \in I'} (S_{i'}, \rho_{i'}). 
\]

disjoint unions of closed hyperbolic surfaces $S_i$ and $S_{i'}$ and multicurves $\rho_i$ and $\rho_{i'}$. Since the cover $p_1 : \tilde{X}_1 \to X_1$ is regular, there exist pairs of closed surfaces and multicurves $(S_g, \gamma_g)$ and $(S_{g'}, \gamma_{g'})$ so that $(S_g, \gamma_g) \cong (S_i, \rho_i)$ and $(S_{g'}, \gamma_{g'}) \cong (S_{i'}, \rho_{i'})$ for all $i \in I$ and $i' \in I'$.

Realize $X_2$ as a union of surfaces and $G_2$ as an amalgamated free product as follows. Since $\tilde{X}_1 \cong \tilde{X}_2$, the space $\tilde{X}_2$ also consists of the surfaces $S$ and $S'$. Let $\gamma_0 : S^1 \to \tilde{X}_1$ be an essential simple closed curve such that $p_1(\gamma_0) = a_1$. Let $S_0 \in S$ and $S_0' \in S'$ be such that $\gamma_0 \subset S_0 \cap S_0'$. Let $S_{g_2} = p_2(S_0)$ and $S_{g_2}' = p_2(S_0')$, and $a_2 = p_2(\gamma_0)$ so that
\[ X_2 = S_{g_2} \cup_{a_2} S'_{g_2}. \] Let \( \gamma_2 : S^1 \to S_{g_2} \) and \( \gamma_2' : S^1 \to S'_{g_2} \) map to the inclusion of \( a_2 \) in \( S_{g_2} \) and \( S'_{g_2} \). Then \( G_2 \cong \pi_1(S_{g_2}) *_{\langle a_2 \rangle} \pi_1(S'_{g_2}) \) as in the statement of the theorem.

Using the regularity of the cover \( p_2 : \tilde{X}_2 \to X_2 \), we construct a covering map \( q_2 : \tilde{X}_2 \to X_2 \) so that \( q_2^{-1}(S_{g_2}, \gamma_2) = S \) and \( q_2^{-1}(S'_{g_2}, \gamma_2') = S' \). We define \( q_2 \) on \( S \) and \( S' \) and prove it is well-defined on the overlap. Let \( f_i : (S_i, \rho_i) \to (S_0, \rho_0) \) and \( f'_i : (S'_i, \rho'_0) \to (S'_0, \rho'_0) \) be homeomorphisms for all \( i \in I \) and \( i' \in I' \). Let \( q_2 = p_2 \circ f_1 \) on \( S \) and let \( q_2 = p_2 \circ f'_{i'} \) on \( S' \). The set \( p_2^{-1}(a_2) \) is a collection of disjoint simple closed curves, each of which projects to \( a_2 \) by the same degree since \( p_2 : \tilde{X}_2 \to X_2 \) is a regular cover. Thus, \( q_2 \) is well defined on the intersection of \( S \) and \( S' \) in \( \tilde{X}_2 \).

Since \( p_1^{-1}(S_{g_1}, \gamma_1) = q_2^{-1}(S_{g_2}, \gamma_2) \) and \( p_1^{-1}(S'_{g_1}, \gamma_1') = q_2^{-1}(S'_{g_2}, \gamma_2') \), claims (b) and (c) follow from Proposition 26.

To prove claim (a), suppose that \( \tilde{X}_1 \) is a \( k_1 \) degree cover of \( X_1 \) and \( \tilde{X}_2 \) is a \( k_2 \) degree cover of \( X_2 \), \( S_g \) is an \( n_i \) degree cover of \( S_{g_1} \), and \( S_{g'} \) is an \( n'_i \) degree cover of \( S'_{g_1} \) for \( i = 1, 2 \). Suppose \( \tilde{X}_1 \) consists of \( m \) copies of \( S_g \) and \( m' \) copies of \( S_{g'} \). Then,

\[
\begin{align*}
k_i &= n_im = n'_im' \\
(6) \quad \frac{k_1}{k_2} &= \frac{n_1}{n_2} = \frac{n'_1}{n'_2}.
\end{align*}
\]

Also, by Lemma 22,

\[
\begin{align*}
\chi(S_g) &= n_1\chi(S_{g_1}) = n_2\chi(S_{g_2}), \\
\chi(S_{g'}) &= n'_1\chi(S'_{g_1}) = n'_2\chi(S'_{g_2}).
\end{align*}
\]

Combining (6) and (7) we have

\[
\frac{\chi(S_{g_1})}{\chi(S_{g_2})} = \frac{\chi(S'_{g_1})}{\chi(S'_{g_2})}.
\]

For the converse, we construct a common cover of \( X_1 \) and \( X_2 \) when the conditions of the theorem hold. Let \( (S_g, \gamma) \) be the cover of \( (S_{g_1}, \gamma_1) \) and \( (S_{g_2}, \gamma_2) \) given by Proposition 26 and let \( (S_{g'}, \gamma') \) be the cover of \( (S'_{g_1}, \gamma'_1) \) and \( (S'_{g_2}, \gamma'_2) \) given by Proposition 26. Let \( L = -\text{lcm}(|\chi(S_{g_1})|, |\chi(S_{g_2})|) \) and let \( L' = -\text{lcm}(|\chi(S'_{g_1})|, |\chi(S'_{g_2})|) \). By construction, \( (S_g, \gamma) \) covers \( (S_{g_1}, \gamma_1) \) by degree \( \frac{2L}{\chi(S_{g_1})} \) and each component of \( \gamma \) covers \( \gamma_1 \) by degree \( \frac{L}{\chi(S_{g_1})} \). Likewise, \( (S_{g'}, \gamma') \) covers \( (S'_{g_1}, \gamma'_1) \) by degree \( \frac{2L}{\chi(S'_{g_1})} \) and each component of \( \gamma' \) covers \( \gamma'_1 \) by degree \( \frac{L'}{\chi(S'_{g_1})} \). Let \( X \) be the space obtained by identifying \( S_g \) and \( S_{g'} \) along the two components of \( \gamma \) and the two components of \( \gamma' \). By condition (a),

\[
L = \frac{L'}{\chi(S_{g_1})} = \frac{L'}{\chi(S'_{g_1})} = n_i.
\]

Thus, \( X \) forms a \( 2n_1 \)-fold cover of \( X_1 \) and a \( 2n_2 \)-fold cover of \( X_2 \). Therefore, \( G_1 \) and \( G_2 \) are abstractly commensurable. \( \square \)
denote the space obtained by identifying curves with 

There exist relatively prime covers 

Suppose We first exhibit a maximal element when the conditions of the proposition hold.

Proof. We first exhibit a maximal element when the conditions of the proposition hold. Suppose \( G \cong \pi_1(S_g) \ast \langle \gamma \rangle \pi_1(S_{g'}) \in \mathcal{C}_S \) where \( \gamma_g \) and \( \gamma_{g'} \) are separating simple closed curves with \( S_g = S_{r,1} \cup_{\gamma_g} S_{s,1} \) and \( S_{g'} = S'_{r',1} \cup_{\gamma_{g'}} S'_{s',1} \) and \( r \neq s \), \( r' \neq s' \).

There exist relatively prime \( p \) and \( q \) and relatively prime \( p' \) and \( q' \) so that

\[
\frac{\chi(S_{r,1})}{\chi(S_{s,1})} = \frac{p}{q} \quad \text{and} \quad \frac{\chi(S'_{r',1})}{\chi(S'_{s',1})} = \frac{p'}{q'}.
\]

So, \( \chi(S_{r,1}) = -dp \), \( \chi(S_{s,1}) = -dq \), \( \chi(S'_{r',1}) = -d'p' \), and \( \chi(S'_{s',1}) = -d'q' \), for some \( d, d' \in \mathbb{N} \). Let \( k = \text{gcd}(d, d') \). Let \( S_{u,1} \) be the surface with one boundary component and Euler characteristic \( -\frac{d}{k}p \), let \( S_{v,1} \) be the surface with one boundary component and Euler characteristic \( -\frac{d}{k}q \), let \( S'_{u',1} \) be the surface with one boundary component and Euler characteristic \( -\frac{d'}{k}p' \), and let \( S'_{v',1} \) be the surface with one boundary component and Euler characteristic \( -\frac{d'}{k}q' \). Let \( S = S_{u,1} \cup_{\gamma} S_{v,1} \) be the surface obtained by identifying \( S_{u,1} \) and \( S_{v,1} \) along their boundary curves and let \( S' = S'_{u',1} \cup_{\gamma'} S'_{v',1} \) be the surface obtained by identifying \( S'_{u',1} \) and \( S'_{v',1} \) along their boundary curves. Then,
\((S_g, \gamma_g)\) covers \((S, \gamma)\) by degree \(k\) and \((S_{g'}, \gamma_{g'})\) covers \((S', \gamma')\) by degree \(k\); so, \(X_g\) covers \(X\) by degree \(k\).

To see that \(\pi_1(X)\) is a maximal element \(C\), let \(H \cong \pi_1(S_h) \ast_{\langle \gamma \rangle} \pi_1(S_{h'}) \in C\) be given by the monomorphisms \(\rho \mapsto [\gamma_h] \in \pi_1(S_h)\) and \(\rho \mapsto [\gamma_{h'}] \in \pi_1(S_{h'})\). By Theorem 27 (b-i) and (b'-i'), \(\gamma_h\) and \(\gamma_{h'}\) are separating simple closed curves so that \(S_h = S_{m,1} \cup_{\gamma_h} S_{n,1}\) and \(S_{h'} = S_{m',1} \cup_{\gamma_{h'}} S_{n',1}\), where

\[
\frac{\chi(S_{m,1})}{\chi(S_{n,1})} = \frac{p}{q} \quad \text{and} \quad \frac{\chi(S_{m',1})}{\chi(S_{n',1})} = \frac{p'}{q'}.
\]

Then \(\chi(S_{m,1}) = -fp\), \(\chi(S_{n,1}) = -fq\), \(\chi(S_{m',1}) = -fp'\), and \(\chi(S_{n',1}) = -fq'\) for some \(f, f' \in \mathbb{N}\). By Theorem 27 (a), \(\frac{\chi(S_{h})}{\chi(S_{g'})} = \frac{\chi(S_{h'})}{\chi(S_{g'})}\), hence \(-\frac{d(p+q)}{d'(p'+q')} = -\frac{f(p+q)}{f'(p'+q')}\), so \(\frac{d}{d'} = \frac{f}{f'}\).

Let \(\ell = \gcd(f, f')\). By a fact of basic number theory, \(\frac{d}{\ell} = \frac{f}{\ell'}\) and \(\frac{d'}{\ell'} = \frac{f'}{\ell}\). Therefore, \((S_h, \gamma_h)\) covers \((S, \gamma)\) by degree \(\ell\) and \((S_{h'}, \gamma_{h'})\) covers \((S', \gamma')\) by degree \(\ell\); thus, \(X_h\) covers \(X\) by degree \(\ell\) as desired.

If \(G \in C_S\) does not satisfy the conditions of the proposition, then there are two groups, \(H_1\) and \(H_2\), in the abstract commensurability class of \(G\) in \(C_S\), where \(H_1 \cong \pi_1(S_{h_1}) \ast_{\langle \gamma \rangle} \pi_1(S_{h_1}')\), \(H_2 \cong \pi_1(S_{h_2}) \ast_{\langle \gamma \rangle} \pi_1(S_{h_2}')\), and, up to relabeling, \(\gamma \mapsto [\gamma_{h_1}] \in \pi_1(S_{h_1})\), \(\rho \mapsto [\gamma_{h_2}] \in \pi_1(S_{h_2})\), where \(\gamma_{h_1}\) is an essential non-separating simple closed curve and \(\gamma_{h_2}\) is a separating simple closed curve. Thus, \((S_{h_1}, \gamma_{h_1})\) and \((S_{h_2}, \gamma_{h_2})\) cannot cover the same pair \((S, \gamma)\), so there is no maximal element in the abstract commensurability class of \(G\) in \(C_S\).

\[\square\]

References


