My research is in the study of hyperbolic dynamical systems defined by geometric objects. As a dynamicist, I am interested in recurrence properties, ergodic geometry, entropy theory, and equilibrium states. As a geometer, I am interested in hyperbolic geometry, three-manifolds, and real projective structures.

Take a closed hyperbolic manifold, say a surface which we denote by $S$. On any such surface, hyperbolic or otherwise, we can define free particle motion, which is more formally known as the geodesic flow. Given a point $p$ and a direction $v \in T^1 S$, the unit tangent bundle to $S$, there is a unique geodesic $\ell_v: \mathbb{R} \rightarrow S$ parameterized at unit speed and based at $p$. We then define the geodesic flow on $T^1 S$ by moving unit tangent vectors along these geodesics at unit speed (see Figure A):

$$\varphi^t: T^1 S \rightarrow T^1 S$$

$$(p, v) \mapsto (\ell_v(t), \frac{d}{dt}\ell_v(t))$$

The geodesic flow of a hyperbolic surface exhibits classical uniformly hyperbolic dynamical behavior. Topologically, the geodesic flow is Anosov, meaning the tangent bundle to $T^1 S$ admits a decomposition into uniformly exponentially contracting and expanding subbundles transverse to the flow direction. Moreover, the geodesic flow satisfies Bowen’s specification property, which has major implications for the complexity of the dynamics. For instance, from transitivity and specification, one has density of periodic orbits, Anosov closing of almost-closed orbits, and topological mixing, meaning any two open sets will eventually and forevermore be mixed together under the geodesic flow.

From the perspective of ergodic theory, the questions of interest are around the probability that orbits exhibit dynamical complexity. A measure of maximal entropy is one which sees the total topological complexity of the dynamical system. Classically, there is a unique measure of maximal for the geodesic flow on a hyperbolic manifold. The existence and uniqueness of a measure of maximal entropy has been extended to manifolds with pinched negative curvature and more recently to rank one manifolds \([12, 15, 13, 14]\).

In my work, I extend classical results to a non-Riemannian three-manifold with isometrically embedded flat tori or Klein-bottles which arises from Benoist’s work on Hilbert geometries \([3]\). These flats are an obstruction to uniformly hyperbolic behavior and prevent regularity of the geodesic flow. The geodesic flow does not preserve a natural volume form on the unit tangent bundle, and it is not differentiable. Nonetheless, morally the geodesic flow should behave similarly to the geodesic flow for a Riemannian analogue. Without tools from Pesin theory, Riemannian rank one manifolds, and Bowen’s equilibrium state program, I am able to extend classical results around geodesic flows to these exotic examples. My results culminate in an explicit construction of the Bowen-Margulis measure, which I prove is an ergodic measure of maximal entropy. In what follows, I present my results around both the ergodic theory and topological dynamics of the system, and I propose a research program for extending my work to a weighted dynamical system.

**Hilbert geometries**

The geometric object that I study is a particular example of a Hilbert geometry. An open set in $n$-dimensional projective space $\mathbb{RP}^n$ is **properly convex** if it can be represented as a bounded convex domain in an affine slice of $\mathbb{R}^{n+1}$. Properly convex domains $\Omega$ admit a Hilbert geometry as follows: For $x, y \in \Omega$, take $a, b$ the intersection points of $\overline{xy}$ with the topological Euclidean boundary $\partial \Omega$ as in Figure A. Define a metric

$$d_\Omega(x, y) := \frac{1}{2} \log |a; x; y; b|$$

where $|a; x; y; b| = \frac{|a-y|}{|a-x|} \frac{|b-x|}{|b-y|}$ denotes the Euclidean cross ratio. Since the cross ratio is a projective invariant, this distance is well defined for our affine representation of $\Omega$. Projective lines are geodesics, but they are not always unique. Note also that $\partial \Omega$ is metrically at infinity.
What is classically studied are the properly convex domains which admit a cocompact action by a discrete group of projective transformations, which are isometries of $d_\Omega$. The first such example is the ellipsoid, which is isometric to $n$-dimensional hyperbolic space when endowed with the Hilbert metric and hence admits cocompact actions by representing $\text{Isom}(\mathbb{H}^n)$ as projective transformations preserving the ellipsoid. The $n$-simplex is isometric to $\mathbb{R}^n$ with a polygonal norm (see Figure A, [11]) and naturally admits a $\mathbb{Z} \times \mathbb{Z}$ action with quotient a torus or Klein-bottle (see Figure B). These particular examples are nothing new. Plenty of nontrivial examples of cocompact actions on properly convex $\Omega$ exist and are originally due to Kac and Vinberg [18].

**The geodesic flow.** Now let $\Omega$ be a properly convex domain in $\mathbb{R}P^n$ and $\Gamma$ a discrete subgroup of projective transformations preserving $\Omega$ such that $\Omega/\Gamma$ is compact. For the quotient manifold $M = \Omega/\Gamma$, we define the geodesic flow on the unit tangent bundle $T^1M$ in the same way as for a hyperbolic manifold by flowing unit tangent vectors along projective lines at unit speed (see the left panel of Figure A). A natural question is: how much of the dynamics of the geodesic flow of hyperbolic manifolds, our first examples of a Hilbert geometry, generalize to other Hilbert geometries, such as those constructed by Kac and Vinberg? Benoist shows that the geodesic flow is Anosov on $T^1M$ (see Figure B). These particular examples are nothing new. Plenty of nontrivial examples of cocompact actions on properly convex $\Omega$ exist and are originally due to Kac and Vinberg [18].

The geodesic flow has the Anosov property if $T_\Omega SM$ admits a decomposition at every $v \in T^1M$ into submanifolds $E^+ \oplus E^- \oplus \mathbb{R}$ such that $E^+$ is contracted and $E^-$ is expanded uniformly exponentially under $\varphi^t$. In the classical hyperbolic case and for a Hilbert geometry with $C^1$-boundary, $E^+$ and $E^-$ project to horospheres passing through the footpoint of $v$ based at $v^+$ and $v^-$, the intersection points with $\partial \Omega$ of the projective line determined by $v$. (see Figure C). A group is $\delta$-hyperbolic if the $\delta$-neighborhood of any two edges of geodesic triangles in the Cayley graph contains the third edge. Then for a cocompact action by isometries, $\delta$-hyperbolicity of $\Gamma$ is equivalent to $\delta$-hyperbolicity of $(\Omega, d_\Omega)$.

**Benoist’s nonstrictly convex three-manifold.** As Hilbert geometries, the Benoist three-manifolds are the first examples of nonsimplicial, nonconvex Hilbert geometries. As dynamical objects, the Benoist three-manifolds are the first example of a compact manifold which is not uniformly hyperbolic, but for which the geodesic flow has hope for hyperbolic behavior. Benoist shows that the quotient must be made up of hyperbolic three-manifold components with boundary flats as codimension one tori or Klein bottles—this structure is known as a JSJ decomposition [3]. The boundary tori and Klein bottles lift to properly embedded triangles, meaning $\Delta \subset \Omega$ and $\partial \Delta \subset \partial \Omega$. The geometry of the projective triangle is natural for the flats in the quotient (see Figure B). The cocompact group action imposes strong geometric rigidity of the universal cover $\Omega$. See Benoist’s webpage for pictures of $\Omega$.

My research is on the geodesic flow for these exotic examples. Because the Benoist three-manifolds are not Riemannian, the geodesic flow does not preserve volume [2] and we cannot apply the classical theory for Riemannian manifolds. Because of the flats, the geodesic flow is not uniformly hyperbolic and it is not differentiable. And lastly, the manifold itself is not CAT(0), so we cannot apply recent results around geodesic flows for CAT(0) spaces [16]. Nonetheless, the flats are codimension 1, so geometrically we should be able to argue that the geodesic flow has a lot of hyperbolic behavior away from the flats. My research provides conclusive evidence for this vaguely presented...
RESEARCH STATEMENT

\[ g = \begin{pmatrix} \lambda^2 & \lambda^{-1} \\ \lambda^2 & \lambda^{-1} \end{pmatrix} \]

\[ h = \begin{pmatrix} \lambda & \frac{1}{\eta} \\ \lambda \eta \end{pmatrix} \]

Figure B. A \( \mathbb{Z} \times \mathbb{Z} \)-action \( \langle h, g \rangle < \text{PSL}(3, \mathbb{R}) \) on the projective triangle, with quotient a torus.

In my recent work, I have proven strong recurrence properties of the geodesic flow of the Benoist three-manifolds. These results establish a platform for studying the ergodic theory of the system.

The length spectrum of a dynamical system is the additive group generated by lengths of periodic orbits for the dynamics. It is classical that an Anosov system with dense length spectrum in \( \mathbb{R}^+ \) is topologically mixing\(^1\). The Anosov property implies the Anosov closing lemma, gluing of nearby orbits, and dense unstable manifolds when paired with density of the length spectrum and topological transitivity. These tools are some of the pieces which go into proving topological mixing. For the Benoist examples, we use the geometry to prove topological transitivity, Anosov closing, and orbit gluing. We then prove density of the hyperbolic length spectrum, which is necessary to prove density of the unstable manifolds since unstable manifolds defined only for hyperbolic orbits. We then conclude topological mixing with the classical proof.

**Theorem** (Anosov Closing, [6]). Let \( \varphi^t : T^1M \to T^1M \) be the geodesic flow of the Benoist examples and \( \Sigma \) the union of flats in \( M \). If for any \( x \in T^1M \setminus T^1\Sigma \) and after a long enough time \( t \), \( \varphi^t x \) is sufficiently close to \( x \), then we can find a nearby closed orbit with period almost \( t \) which follows the orbit of \( x \) for time \( t \).

Note that the required closeness and wait time to find a nearby closed orbit is not uniform over \( x \).

**Theorem** ([6]). The geodesic flow of the Benoist examples is topologically mixing.

**Entropy.** In his thesis, Crampon proved a strong rigidity result for the entropy of strictly convex divisible Hilbert geometries. His result confirms that nontrivial strictly convex Hilbert geometries are indeed topologically distinguished from the Riemannian case via topological entropy: for \( \varphi^t \) the geodesic flow on \( T^1M \), where \( M \) is the quotient of a strictly convex Hilbert geometry, \( h_{\text{top}}(\varphi) \leq n - 1 \), with equality only for Riemannian hyperbolic case.

Crampon’s techniques depend on the Anosov property and require the differential of the geodesic flow to compute Lyapunov exponents. Even though the topological entropy of the geodesic flow for the Benoist examples should satisfy the same inequality, the theorem does not extend. In my work, I prove topological phenomena that are weaker analogues of the expansivity and specification properties, the latter of which was introduced by Bowen and ultimately has powerful implications for the ergodic theory of the system [5]. I first prove entropy-expansiveness (see [4]) for any Hilbert geometry, and then I prove a weak-specification property for the Benoist examples. This classical proposition follows.

**Proposition** (Characterizations of entropy, [6]). For the geodesic flow of the Benoist examples, we have

\[ \rho(\varphi) = h_{\text{top}}(\varphi) = h_{\text{vol}}(\Omega) \]

Here, \( h_{\text{vol}} \) is the exponential growth rate of the volume of metric balls in \( \Omega \), the universal cover of the Benoist examples, and \( \rho(\varphi) \) is the exponential growth rate of isolated periodic orbits of the geodesic flow. A first consequence of this proposition is that \( h_{\text{top}} = \rho > 0 \).

\(^1\) The geodesic flow is topologically mixing if for all \( U, V \) open, there exists a \( T \in \mathbb{R} \) such that \( t \geq T \Rightarrow \varphi^t U \cap V \neq \emptyset \).
Figure C. The stable manifold $W^{ss}(v)$ is contracted under the Hilbert geodesic flow of $\Omega$ with $C^1$ boundary. Note that $W^{ss}(v)$ projects to the horosphere at $v^+$ passing through $\pi v$. Stable manifolds can be defined for $v$ if $\Omega$ admits a unique supporting hyperplane at $v^+$.

**Ergodic theory**

The variational principle states that for any dynamical system $f$ on a measure space $X$, if $\mathcal{M}$ is the collection of $f$-invariant probability measures on $X$, then

$$h_{top}(f) = \sup_{m \in \mathcal{M}} h_m$$

where $h_m$ is the measure theoretic entropy of $m$ for the dynamics of $f$ on $X$. A measure of maximal entropy is an invariant measure which realizes the supremum. For symmetric spaces, the measure of maximal entropy is volume measure or Liouville measure. Volume cannot be a measure of maximal entropy for the geodesic flow of the Benoist examples, since volume is not flow invariant. Thus, we need to construct an alternate candidate for a measure of maximal entropy.

Patterson-Sullivan theory is a classical mechanism for constructing a measure of maximal entropy called the **Bowen-Margulis measure** following the independent work of Bowen and Margulis on the geodesic flow of hyperbolic manifolds. Patterson-Sullivan theory has been applied to many flavors of the geodesic flow, such as for geometrically finite hyperbolic manifolds, rank one manifolds, and CAT(0)-spaces [17, 13, 14, 16]. Given the irregular behavior of the geodesic flow of the Benoist examples, some elegant proofs from Patterson-Sullivan theory require careful attention and at times alternative strategies.

**Theorem** ([6]). The exists a family of measures defined on the visual boundary of universal cover $\Omega$ parameterized by points $x \notin \Sigma$, the union of flats in $\Omega$. This family of measures, classically known as the Patterson-Sullivan density, has dimension $h_{top}(\varphi)$. The Patterson-Sullivan density is the unique such family of measures with dimension $h_{top}(\varphi)$, and assigns zero-measure to the boundary of flats.

A corollary of this theorem is sharp asymptotics for sphere volume growth, and as a corollary we have that $\Gamma$ is divergent.

**Bowen-Margulis measure.** Using the Patterson-Sullivan measures at infinity, we construct the Bowen-Margulis measure. Given a Patterson-Sullivan measure $\mu_\infty$ on $\partial_1 \Omega$, we identify a projective line with a pair of points in the boundary. Then $\mu_\varphi = \mu_\infty \otimes \mu_\infty$ is a finite measure on the space of projective lines of $\Omega$. We then normalize the measure in such a way that $\mu_\varphi$ is $\Gamma$-invariant, and hence induces a measure on the projective lines of $M = \Omega/\Gamma$, called $\tilde{\mu}_\varphi$. This measure $\tilde{\mu}_\varphi$ induces a $\varphi^t$-invariant measure on $SM$. (cf. [17]).

We apply the Hopf argument to prove mixing using Babillot’s method [1] and density of the hyperbolic length spectrum, proven in the topological study of the system:

**Theorem** (Mixing of the Bowen-Margulis measure, [6]). The Bowen-Margulis measure is not only ergodic but also mixing for the geodesic flow of the Benoist examples: for all Borel measurable $A, B \subset T^1 M$,

$$\mu_{BM}(A \cap \varphi^t(B)) \xrightarrow{t \to \pm \infty} \mu_{BM}(A)\mu_{BM}(B)$$

I am currently showing that the Bowen-Margulis measure is not only a measure of maximal entropy, it is the unique such measure. I am extending the work of Knieper for rank one manifolds [14], which follows naturally from the
tools I have built thus far. My work thus extends the classical theory to a system which lacks geometric and analytic regularity.

**Theorem** ([6]). The Bowen-Margulis measure is the unique measure of maximal entropy for the geodesic flow of the Benoist examples.

**Equilibrium states**

A project that follows my current work is generalizing the entropy theory for the Benoist examples to the theory of equilibrium states. A weight $F: T^1M \to \mathbb{R}$ called a potential function is added to the dynamical system. Then the topological pressure of $(\varphi, F)$ is a natural generalization of topological entropy, and satisfies the following variational principle:

$$P_{top}(\varphi, F) = \sup_{m} \left( h_m(\varphi) + \int_{T^1M} F \, dm \right)$$

A measure for which the supremum is realized is an equilibrium state for the potential function $F$.

**The general theory.** The natural interesting questions in the theory of equilibrium states take a similar form to those for entropy theory. Do equilibrium states exist for potential functions with sufficient regularity? Gibbs measures, denoted $m_F$, are the equilibrium states that generalize Bowen-Margulis measure. Existence and uniqueness is known in many classical cases, originating with Bowen as an application of the specification property [4].

The geodesic flow of the Benoist examples does not admit direct application of an existing theory. The flow is too irregular and satisfies a version of specification which is weaker than what is needed for Bowen’s techniques. The Benoist manifolds lack the curvature needed to apply work of Paulin, Pollicott, and Schapira for manifolds with pinched negative curvature [15].

Recently, Climenhaga and Thompson [7] have extended Bowen’s results on equilibrium states for systems which satisfy a weaker, nonuniform version of specification. These results only require continuity of the flow. Therefore, a study of existence and uniqueness of Gibbs states for the geodesic flow of the Benoist examples would be an interesting and promising future project.

**Sinai-Ruelle-Bowen measure.** Another interesting future project is the study of a strong unstable potential for the Benoist examples. In the Riemannian case, the strong unstable potential is defined to be:

$$F^{su}(v) = -\frac{d}{dt} \bigg|_{t=0} \log \text{Jac} \left( \varphi^t|_{W^{su} v} \right) (v)$$

For $C^1$ geodesic flows, the strong unstable potential captures the pointwise exponential growth rate of the differential of the geodesic flow restricted to the strong unstable manifold. If $M$ is a compact manifold of variable negative curvature, the natural analogue $\lambda$ for volume measure is the Gibbs measure for $F^{su}$ [15]. This measure $\lambda$ is natural in the sense that the Birkhoff averages for $\lambda$ converge Lebesgue-almost everywhere. Such a measure $\lambda$ is called the Sinai-Ruelle-Bowen (SRB) measure. Examples when SRB is nontrivial include strange attractors such as the Lorenz attractor. Studying equilibrium states for the potential function $F^{su}$ when volume is not invariant recovers SRB measure, a natural alternative for Lebesgue measure.

The Benoist examples set up a particularly enticing case study of SRB measure. Volume is not invariant for the geodesic flow, so we are guaranteed a nontrivial SRB measure. Unlike for the Riemannian rank one case, which is still an active object of study for thermodynamic formalism, the geodesic flow is not differentiable, so $F^{su}$ is not well-defined at first glance. Is it possible to define $F^{su}$ for a set of large measure, or construct an appropriate geometric analogue for $F^{su}$? If we can construct the strong unstable potential such that the equilibrium state for $F^{su}$ is an SRB measure, then we expect there should be a distinct SRB measure realized as an equilibrium state for the analogous strong stable potential. Such a result would extend Crampon’s work for the strictly convex case [9].

These questions on equilibrium states and SRB measures are the foundation for my ongoing study of nonuniformly hyperbolic geodesic flows.

**References**
