Acylindrical Hyperbolicity and CAT(0) Groups

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Consider a discrete group G which acts on a $\delta-$hyperbolic space X by isometries. We can classify these group elements into three types:

- Elliptic
- Loxodromic
- Parabolic*

Arguably, the most natural generalzation of hyperbolicity is the notion of relative hyperbolicity, which does admit parabolic actions. An illustrative example is

$$F_2 \cong \pi_1(\Sigma_{1,1}) \curvearrowright \mathbb{H}^2$$

Definition

An metric space action $G \curvearrowright X$ *is called acylindrical if for every* $\epsilon > 0$ *there exist* $R, N > 0$ *such that for any two points* x, y *such that* $d(x, y) \geq R$*, the set*

$$\{g \in G | d(x, g.x) \leq \epsilon, d(y, g.y) \leq \epsilon\}$$

has cardinality less than N*.*

Definition

A group is called **acylindrically hyperbolic** *if it acts nonelementarily and acylindrically on a hyperbolic space*

Immediately we see hyperbolic and relatively hyperbolic groups meet this criterion, so we may think of acylindrical hyperbolicity as a further generalization on relative hyperbolicity.

Acylindrically hyperbolic groups is a respectably wide class, including:

- 'Most' Mapping Class Groups
- $\mathrm{Out}(F_n)$ for $n \geq 2$
- Indecomposable RAAGs
- 1 Relator, $\geq$3 Generator Groups
- 'Most' 3-Manifold Groups

Definition

For an acylindrically hyperbolic group G, we call an element $g \in G$ a **generalized loxodromic** *if it acts as a loxodromic for some acylindrical action on a hyperbolic space*

It is possible for a group element to act loxodromically for some acylindrical action on a hyperbolic space, but elliptically for another. In fact, the existence of such 'universal actions', where each generalized loxodromic acts as such on the same space, is not guaranteed for all A.H. groups.

So what do these elements look like?

- Pseudo-Anosovs in MCGs
- Loxodromics in Relatively Hyperbolic Groups
- Generally ‘negatively curved directions’

Some versions of negative curvature in spaces:

- Contracting rays
- Morse geodesics
- Superlinear/Quadratic/Exponential Divergence

$$\textit{Contracting} \implies \textit{Gen.Lox.} \implies \textit{Morse}$$

Definition

A CAT(0) space is a metric space in which geodesic triangles are no 'fatter' than those in Euclidean space. A CAT(0) group is a group acting geometrically on a CAT(0) space.

Definition

Geodesics that do not bound a half-flat in CAT(0) spaces are called **Rank One** *geodesics. Isometries of these spaces which have axes satisfying this property are called* **Rank One** *elements*

Theorem (Charney)

In a CAT(0) space, the following are equivalent for geodesics:

- *Rank One*
- *Contracting*
- *Morse*
- *At least quadratic divergence*

Conjecture (Rank Rigidity)

Let X be a CAT(0) space and Γ be a group acting on it geometrically. Then if X is not a Euclidean Building or a 'higher rank symmetric space', Γ must contain a rank one isometry.

Ex. 2-dimensional complexes [Ballmann, Brin], Cube Complexes [Caprace, Sageev]

Thank you for listening!

Questions?